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Finite element analysis of two-dimensional EM scattering via Padé approximation for complex permittivity

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[1] We present a computationally efficient algorithm which combines the finite element method with Padé approximation. The combination is used to solve the problem of transverse magnetic and transverse electric scattering from homogeneous lossy dielectric cylinders over a continuous range of the complex permittivity variable by requiring the finite element solution only at a single complex permittivity value. The proposed method is based on (1) assuming a power series expansion for the unknown solution vector, the excitation vector, and the system matrix, (2) substituting this into the system matrix equation, and (3) finding the recursion relation for the solution vectors. *INDEX TERMS:* 0669 Electromagnetics: Scattering and diffraction; 0644 Electromagnetics: Numerical methods; *KEYWORDS:* Padé approximation, finite element method, electromagnetic wave scattering

1. Introduction

[2] Several frequency domain methods are available in the literature for the solution of electromagnetic scattering by dielectric cylinders of arbitrary cross section [*Abu-Zaid et al.*, 1999; *Peterson and Castillo*, 1989]. One of the disadvantages of these frequency domain techniques, however, is the computational cost involved in getting the solutions over an interval of a predefined parameter, such as frequency or permittivity. It is required to calculate the fields for each such distinct parameter value to obtain the complete behavior over the interval, and in order to get an accurate representation of the overall response, one needs to repeat the calculations at finer increments of complex permittivity (or frequency). This can be computationally intensive, and the total CPU time needed to compute the fields can be highly prohibitive. To overcome this problem, we replace the unknown solution vector with its complex permittivity (or frequency) power series, and then suitable Padé approximants are obtained from the coefficients of the power series [*Baker and Graves-Morris*, 1996]. The method was originally developed to calculate frequency response [*Jiao et al.*, 1999; *Kuzuoglu and Mittra*, 1999; *Gong and Volakis*, 1996; *Zhang and Jin*, 1998]. It has been observed that such an approximation provides an output response that is highly accurate over a wide range by solving the scattering problem only at a single complex permittivity (or frequency) value. In this paper, we will be interested only in complex permittivity changes at a fixed frequency. A following paper will involve frequency variations at constant complex permittivity. Work is also in progress for both permittivity and frequency variations in their respective intervals.

2. Formulation

[3] Consider an infinitely long homogenous and lossy dielectric cylinder with an arbitrary cross section. Either a transverse magnetic (TM) or a transverse electric (TE)-polarized plane wave is incident on the scatterer at an angle φ^{inc} with respect to the $+x$ axis. The situation is depicted in Figure 1. Based on the total field formulation, the scalar wave equation is written as [*Peterson and Castillo*, 1989; *Jin*, 1993]

$$\text{TM polarization}\quad \nabla\cdot\left(\frac{1}{\mu_r}\nabla E_z\right)+k_o^2\varepsilon_r' E_z=0,$$

$$\text{TE polarization}\quad \nabla\cdot\left(\frac{1}{\varepsilon_r'}\nabla H_z\right)+k_o^2\mu_r H_z=0,\qquad (1)$$

where $\Omega=\Omega_1\cup\Omega_2$, $(x,y)\in\Omega$, ε_r' is the complex relative permittivity of the scatterer, k_o is the free space wave

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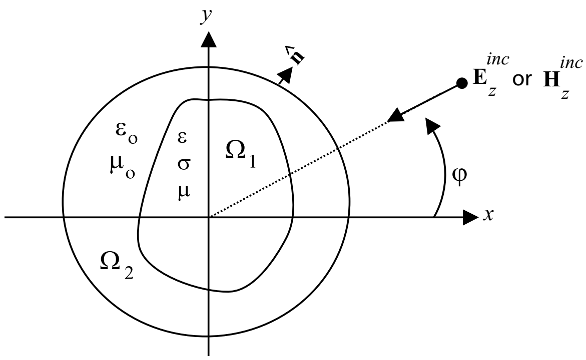

Figure 1. Cross section of the cylindrical geometry under consideration.

number, and μ_r is the relative permeability of the scatterer. Multiplying (1) by a weighting function $W(x, y)$, integrating over the domain Ω, and then using some identities yields the following weak equations [*Peterson and Castillo*, 1989; *Jin*, 1993]:

$$\text{TM polarization} \iint_{\Omega_1} \left[-\nabla W \cdot \nabla E_z + k_0^2 \mu_r \varepsilon_r' W E_z\right] d\Omega_1 + \mu_r \iint_{\Omega_2} \left[-\nabla W \cdot \nabla E_z + k_o^2 W E_z\right] d\Omega_2 + \mu_r \int_{\partial\Omega} W\left(\nabla E_z \cdot \hat{\mathbf{n}}\right) dl = 0, \quad (2a)$$

$$\text{TE polarization} \iint_{\Omega_1} \left[-\nabla W \cdot \nabla H_z + k_0^2 \mu_r \varepsilon_r' W H_z\right] d\Omega_1 + \varepsilon_r' \iint_{\Omega_2} \left[-\nabla W \cdot \nabla H_z + k_o^2 W H_z\right] d\Omega_2 + \varepsilon_r' \int_{\partial\Omega} W\left(\nabla H_z \cdot \hat{\mathbf{n}}\right) dl = 0, \quad (2b)$$

The line integrals on $\partial\Omega$ are readily obtained by means of the absorbing boundary conditions [*Peterson and Castillo*, 1989]. In the context of finite element method (FEM) discretization of (2a) and (2b), there results a matrix equation of the form [*Peterson and Castillo*, 1989; *Jin*, 1993]

$$\mathbf{K}\left(\varepsilon_r'\right)\mathbf{C}\left(\varepsilon_r'\right) = \mathbf{b}\left(\varepsilon_r'\right), \quad (3)$$

where $\mathbf{K}(\varepsilon_r')$ is an $N \times N$ complex matrix, $\mathbf{C}(\varepsilon_r')$ is the $N \times 1$ unknown solution vector, $\mathbf{b}(\varepsilon_r')$ is the N excitation vector, and N is the total number of nodes in the grid. The problem of interest here is to find $\mathbf{C}(\varepsilon_r')$ for a finite range of complex permittivity values by solving (3) only at a single complex permittivity value, say ε_{rc}'. To accomplish this task, let us start by assuming that the unknown vector $\mathbf{C}(\varepsilon_r')$ has a power series expansion about ε_{rc}' of the form

$$\mathbf{C}\left(\varepsilon_r'\right) = \sum_{i=0}^{\infty} \mathbf{c}_i \left(\varepsilon_r' - \varepsilon_{rc}'\right)^i. \quad (4)$$

Then our immediate goal is to derive recursive formulas for the unknown expansion vectors $\mathbf{c}_i$. Noting that the system matrix $\mathbf{K}$ can be written as a finite polynomial in ε_r', it is easy to obtain its power series expansion as

$$\mathbf{K}\left(\varepsilon_r'\right) = \sum_{i=0}^{1} \mathbf{D}_i \left(\varepsilon_r' - \varepsilon_{rc}'\right)^i. \quad (5)$$

Similarly, the right-hand-side vector, namely $\mathbf{b}$, is also expanded as

$$\mathbf{b}\left(\varepsilon_r'\right) = \sum_{i=0}^{\delta_p} \mathbf{F}_i \left(\varepsilon_r' - \varepsilon_{rc}'\right)^i, \quad (6)$$

$$\delta_p = \begin{cases} 0 & \text{for TM polarization} \\ 1 & \text{for TE polarization} \end{cases},$$

where $\mathbf{D}_i$ and $\mathbf{F}_i$ are known expansion matrices and vectors, respectively. Substitution of (4), (5), and (6) into (3) and left multiplication by $\mathbf{D}_o^{-1}$ gives

$$\left[\mathbf{I} + \left(\varepsilon_r' - \varepsilon_{rc}'\right)\mathbf{D}_o^{-1}\mathbf{D}_1\right]\left[\sum_{i=0}^{\infty} \mathbf{c}_i \left(\varepsilon_r' - \varepsilon_{rc}'\right)^i\right] = \mathbf{D}_o^{-1} \sum_{i=0}^{\delta_p} \mathbf{F}_i \left(\varepsilon_r' - \varepsilon_{rc}'\right)^i. \quad (7)$$

Expanding (7) and equating terms of similar powers, we obtain the recursive formula

$$\begin{aligned} \mathbf{c}_o &= \mathbf{D}_o^{-1}\mathbf{F}_o & i &= 0, \\ \mathbf{c}_i &= \mathbf{D}_o^{-1}\mathbf{F}_1\delta_p - \mathbf{D}_o^{-1}\mathbf{D}_1\mathbf{c}_{i-1} & i &\geq 1. \end{aligned} \quad (8)$$

[4] A technique that extends the convergence range of the power series expansion is the so-called Padé approximation, which allows us to obtain a representation of $\mathbf{C}(\varepsilon_r')$ which is valid over a much wider range of complex permittivity than does the power series. Once the power series coefficient vectors $\mathbf{c}_i$ are obtained, Padé approximants are found according to what follows. Consider a single component of the vector $\mathbf{C}(\varepsilon_r')$ and denote it by $C^j(\varepsilon_r')$. If we express $C^j(\varepsilon_r')$ by its truncated power series around ε_{rc}', then the $[L/M]$ Padé

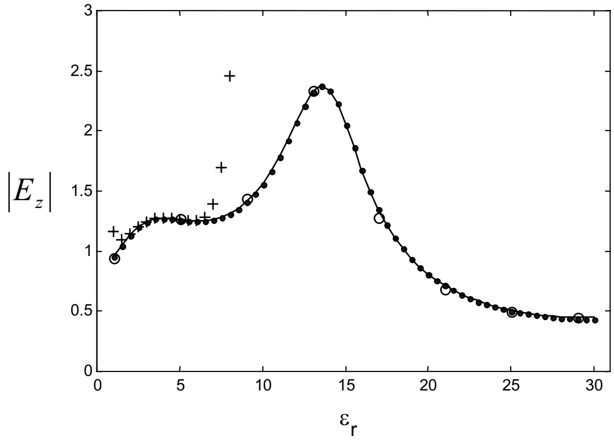

Figure 2. Amplitude of electric field versus relative permittivity at $(x, y) = (-0.1, 0)$ for a circular cylinder. FEM, solid circles; exact Mie series solution, open circles; Padé approximation, solid curve; power series, pluses .

approximants are obtained by representing $C^j(\varepsilon_r')$ as the quotient of two polynomials [*Baker and Graves-Morris*, 1996]

$$C^j(\varepsilon_r') \approx \sum_{i=0}^{N} c_i^j (\varepsilon_r' - \varepsilon_{rc}')^i = \frac{\sum_{i=0}^{L} p_i (\varepsilon_r' - \varepsilon_{rc}')^i}{\sum_{i=0}^{M} q_i (\varepsilon_r' - \varepsilon_{rc}')^i} = \frac{P_L}{Q_M}, \tag{9}$$

where c_i^j is the jth component of the ith power series coefficient vector $\mathbf{c}_i$, $Q_M(0) \equiv 1$, and $N = M + L$. Upon expanding (9), we obtain $M + N + 1$ equations. From the last M equations we solve for the q, specifically,

$$\begin{bmatrix} c_L^j & c_{L-1}^j & \cdots & c_{L-M+1}^j \\ c_{L+1}^j & c_L{}^j & \cdots & c_{L-M}^j \\ \vdots & \vdots & & \vdots \\ c_{L+M-1}^j & c_{L+M-2}^j & \cdots & c_L^j \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_M \end{bmatrix} = - \begin{bmatrix} c_{L+1}^j \\ c_{L+2}^j \\ \vdots \\ c_{L+M}^j \end{bmatrix}. \tag{10}$$

Then from the first $L + 1$ equations, the p are found by a simple matrix multiplication as

$$\begin{bmatrix} c_o^j & 0 & \cdots & 0 \\ c_1^j & c_o^j & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ c_L^j & c_{L-1}^j & \cdots & c_o^j \end{bmatrix} \begin{bmatrix} 1 \\ q_1 \\ \vdots \\ q_L \end{bmatrix} = - \begin{bmatrix} p_o \\ p_1 \\ \vdots \\ p_L \end{bmatrix}. \tag{11}$$

3. Numerical Results

[5] To demonstrate the efficiency of the method, a number of examples are carried out on a 350 MHz Pentium II processor and 64 Mb memory personal computer. The computer code employs isoparametric eight-noded quadrilateral elements and numerical integration for double and line integrals. In all examples, the grid is truncated by a second-order circular absorbing boundary of radius r_o

[6] As a first check, we consider a dielectric circular cylinder illuminated by a 300 MHz TM-polarized plane wave with zero incidence angle. The radius of the cylinder is $0.1\lambda_o$, the total number of nodes is 112, $\mu_r = 1$, and $r_o = 0.2\lambda_o$. The problem is solved around the expansion point $\varepsilon_{rc}' = 4 - j0.27$ with [3/4] Padé approximants combined with FEM; it requires 22.19 s to obtain the solution. The problem is solved directly with FEM by changing ε_r from 1 to 30 in steps of 0.5 (i.e., 59 real permittivity points) while fixing the imaginary part ($\varepsilon_r' = \varepsilon_r - j0.27$); the solution is obtained in 957.85 s. The results are compared with the exact Mie series solution as shown in Figures 2 and 3. The same circular scatterer is illuminated by a 300 MHz TE-polarized plane wave, and in order to have a better idea about the computation time, the number of nodes is increased to 344. The expansion point is taken as $\varepsilon_{rc}' = 15 - j0.06$. The [3/3] Padé approximation combined with FEM requires 82.8 s, while the direct FEM solution with ε_r stepped from 1 to 30 in increments of 0.5 ($\varepsilon_r' = \varepsilon_r - j0.06$) consumes 3029.7 s. The result is compared with exact Mie series and shown in Figure 4.

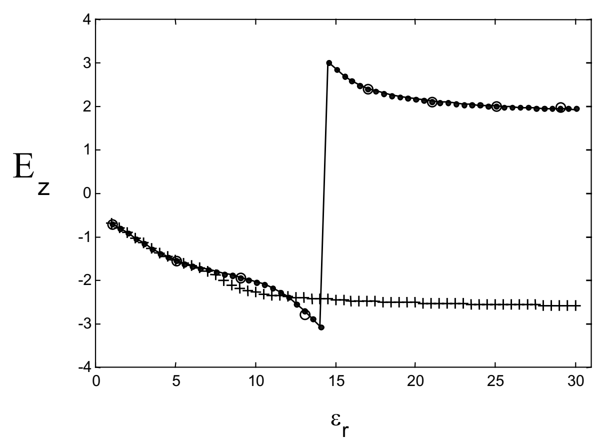

Figure 3. Phase of electric field versus relative permittivity at $(x, y) = (-0.1, 0)$ for a circular cylinder. Symbols are as in Figure 2.

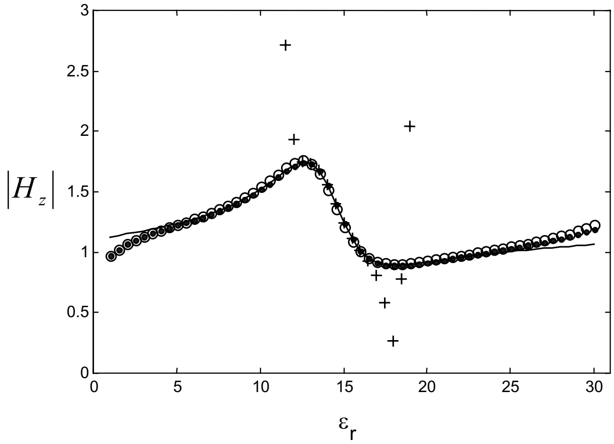

Figure 4. Amplitude of magnetic field versus relative permittivity at $(x, y) = (-0.1, 0)$ for a circular cylinder. Symbols are as in Figure 2.

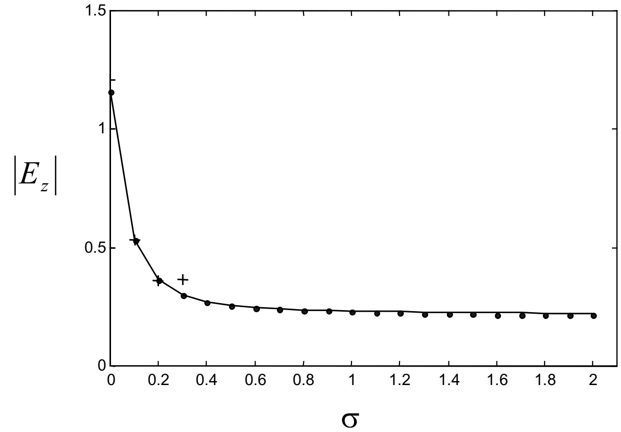

Figure 6. Amplitude of electric field versus conductivity at $(x, y) = (-0.2, 0)$ for a rectangular cylinder. Symbols are as in Figure 5.

[7] To extend the validity of the technique to non-circular scatterers, we considered an ellipse with $0.4\lambda_o$ major axis and $0.2\lambda_o$ minor axis, $r_o = 0.3\lambda_o$ and $\varepsilon'_{rc} = 11 - j0.3$; the grid contains a total of 769 unknowns. The excitation is a TM-polarized plane wave at 600 MHz with zero incidence angle. The FEM combined with [4/4] Padé approximation requires 218.93 s. A direct FEM solution is obtained by changing ε_r from 1 to 25 by 0.5 increments in 5757.6 s. The field magnitude is shown in Figure 5.

[8] In addition to expanding about the real part of ε'_r and fixing the imaginary part, we choose in this last example to expand about the conductivity σ of the scatterer; that is, we fix the real part of ε'_r and change the imaginary part. To illustrate, a rectangular scatterer of $0.2\lambda_o$ length along the x axis and $0.4\lambda_o$ length along the y axis is considered, with $r_o = 0.3\lambda_o$ and a 300 MHz TM-polarized plane wave impinging from $\varphi^{inc} = 0$. The expansion point is chosen as $\varepsilon'_{rc} = 4 - j6$ which is equivalent to a conductivity of 0.1; the single-point [3/3] Padé approximants combined with FEM require 207.56 s for a total of 769 unknowns. The direct FEM solution is obtained by stepping the conductivity from 0 to 2 in steps of 0.1 (21 conductivity values), which corresponds to changing the imaginary part of ε'_r from 0 to 120. The solution is obtained in 2525.9 s. The result is shown in Figure 6.

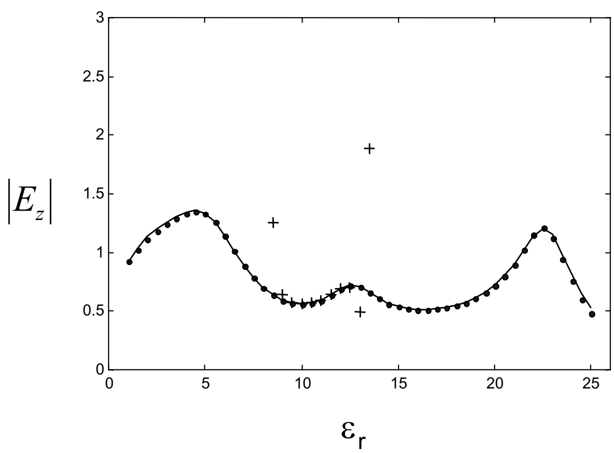

Figure 5. Amplitude of electric field versus relative permittivity at $(x, y) = (-0.3, 0)$ for an elliptic cylinder. FEM, solid circles; Padé approximation, solid curve; power series, pluses.

4. Concluding Remarks

[9] FEM by itself is a very powerful solution method for its capability to handle arbitrarily shaped geometries. Its computational efficiency increases considerably, however, when its combined with Padé approximation technique. This is obvious from the examples considered above. This combination provides an accurate and economical way for solving two-dimensional electromagnetic wave scattering problems over a wide range of complex permittivity values. The beauty of the technique is that it requires inverting a sparse matrix only once, so the overall cost in CPU time is extremely reduced. It is needless to mention that Padé approximation provides a much wider range of convergence over the power series. The small increase in memory

requirements is not a real drawback of the method, because of the fact that all matrices involved in the calculations are largely sparse, which is a distinguishing feature of FEM. No serious attempt was made to decide which type of Padé approximants is better, but it seems that the diagonal approximants ($L = M$) give the best possible approximation.

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