An - Najah Univ. J. Res.(N. Sc.) Vol. 33(1), 2019

An Optimal Control for A Homogeneous Linear Dynamical Systems

التحكم الامثل للانظمة الديناميكية الخطية المتجانسة

Adnan Daraghmeh*
عدنان دراغمة
Department of Mathematics, Faculty of Sciences, An-Najah National University, Nablus, Palestine
* Corresponding Author: adn.daraghmeh@najah.edu
Received: (19/9/2018), Accepted: (6/12/2018)

Abstract

In this article we introduce an approach to find the optimal control for homogeneous linear dynamical systems. We introduce the linear quadratic regulator problem for the reduced order model of a dynamical system. We apply the singular perturbation approximations techniques to the original system and obtained our reduced order system. In this paper, our goal is to examine the solution of the problem as the positive parameter ε tends to zero. Our approach that is based on a formal asymptotic expansion of an algebraic Riccati equations associated with the Pontryagin maximum principle. We derive an approximate expressions for the optimal feedback controls. We compute the L_2-norm beween the optimal control of the original and reduced system and compare it with the L_2-norm of the solution of the Riccati equations of the original and reduced systems. We test the results numerically and obtained a good closed-loop performance for the model reduction.

Keywords: Optimal control, Riccati equation , Homogeneous systems, Closed-loop systems.

ملخص

في هذا العمل ، قمنا بدراسة الانظمة الخطية المتجانسة والهدف العام ايجاد التحكم الامثل لها. قمنا باستخدام احدى الطرق المتبعة في نظرية التحكم

لايجاد نظام مصغر عن النظام الاصلي. قمنا بدراسة معادلة ريكاتي الجبرية لكل من النظامين الاصلي والمصغر وكذلك استخدام مبدا بنترياجين المطلق. قمنا باشتقاق معادلات جبرية للتحكم الامثل لكل من النظامين الاصلي والمصغر. لاختابر هذه النتائج التي توصلنا اليها ، قمنا بتطبيق مثال عددي وقياس الفرق بين التحكم الامثل للنظامين

الكلمات المفتاحية : التحكم الامثل ، معادلة ريكاتي ، الانظمة المتجانسة ، الانظمة المغلقة.

1 Introduction

The Linear Quadratic Regulator (LQR) is a special case of optimal control problem that can be solved analytically.
It is a method used abundantly to design linear controllers for linear system which possesses suitable robustness with minimum gain margin-6db and maximum gain margin to infinity and the same it can reach to 60 degree of phase margin. The design parameters for LQR are the wieghting matrices in the objective function and should be selected by the designer. Since thses matrices directly affect the optimal control performance many discussions have been done to select these matrices called eigen-structure assignment (F.L.Lewis & V.L.Syrmos, 1995; G.P.Liu & R.J.Patton, 1998; J.W.Choi & Y.B.Seo, 1998). In addition to being helpful in design procedures, the singular perturbation approach is an indispensable tool for analytical investigations of robustness of system properties, behavior of optimal controls near singular arcs, and other effects of intentional or unintentional changes of system order(Kokotovic, O'malley, & Sannuti, 1976).

The linear quadratic regulator (LQR) defines an important class of control problems which involve linear dynamics and a quadratic cost (Kokotovic, 1984).

Optimal control deals with the problem of finding a control for a given system such that a certain optimality criterion is achieved.

The optimization technique will try to determine the optimal state trajectory and input signal that minimizes the performance index subject to the constrains imposed by the state equation and the initial state (O'Malley, 1972).
Necessary conditions that must be satisfied by the optimal solution can be derived from Hamiltonian function (O'Malley Jr, 1975; G.P.Liu & R.J.Patton, 1998).
For finite time-horizon optimal problems, one of the most actively investigated singularly perturbed optimal control problems is the linear quadratic regulator problems. Most of these approaches are based on the singularly perturbed differential Riccati equation. An alternative approach via boundary value problems is presented in (O'Malley, 1972). Its relationship with the Riccati aproach is analyzed in (O'Malley Jr, 1975).
One of the basic results of control theory is the solution of the optimal linear state regulator problem by Kalman (1960), which reduces the problem to the solution of a matrix Riccati equation (Kokotovic, 1984).

In this article the LQR is used to find an optimal control that minimizes the quadratic cost function. To do that we have used formal asymptotics for the Pontryagin maximum principle (PMP) and the underlying algebraic Riccati equation. The outcome of this work are case description under which balanced truncation and the singular perturbation approximation give good closed-loop performance. The formal calculations are validated by numerical experiment illustrating that the reduced-order can be used to approximate the optimal control of the original system.
This paper is organized as follows:
In Section (2), some preliminaries for linear dynamical system are presented. In sections (3) and (4) we establish different types of singular perturbation requlator problems. To show the effectiveness of the proposed methods, numerical example is given in section (5). Finally, we make some conclusion remarks in section (6).

2 Preliminaries

We start by considering the following continuous linear dynamical system defined as:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \\ x(0) &= x_0 \end{aligned} \tag{2.1}$$

where $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times n}$ and $D^{p\times m}$ are constant matrices and x, u are the state and the input of the system respictively and $x(0)$ represents the initial condition .
We assume that the linear system described by equation (2.1) is controllable and observable.
The quadratic cost function J is defined by the following equation :

$$J = \frac{1}{2}\int_0^\infty (x^T Qx + u^T Ru)dt \tag{2.2}$$

where $Q = C^T C \geq 0$ is a positive semi definite matrix representing the cost penalty of the states and $R > 0$ is a positive definite matrix that represents the cost penalty of the input.
Our gaol is to find the optimal control u that minimizes the quadratic cost function J in (2.2) subject to the constraint

$$\dot{x} = Ax + Bu$$

We denote the optimal control by u^* such that the following is hold:

$$J(u^*) \leq J(u), \quad \forall u \in L^2$$

and

$$\dot{x} = Ax + Bu^*$$

with optimal solution denoted by x^*.
The next step is to introduce an approach that depends on the Hamiltonian function defined in the following form:

$$H = \frac{1}{2}\left(x^T Qx + u^T Ru\right) + \lambda^T (Ax + Bu) \tag{2.3}$$

where $\lambda \in \Re^n$ is called the co state variable.
The following theorem describes the way in which we can find the optimal control that minimizes the quadratic cost function J in equation (2.2).

Theorem 2.1. *(Murray, 2009; Knowles, 1981)(Maximum Principle) If x^*,u^* is optimal (or a solution of the LQR), then there exists a solution $\lambda^* \in \Re^n$ such that:*

$$\dot{x} = \frac{\partial H}{\partial \lambda} \tag{2.4}$$

$$\dot{\lambda} = -\frac{\partial H}{\partial x} \tag{2.5}$$

and the minimality condition of the Hamiltonian

$$H(x^*, u^*, \lambda^*) \leq H(x^*, u, \lambda^*)$$

holds for all $u \in \Re^m$

For more details on the proof (see (Murray, 2009; Knowles, 1981)).
If H is a differentiable function, then to minimize H with respect to u we can find our optimal control input.
The following condition must be true to find such u that is:

$$\frac{\partial H}{\partial u} = 0 \tag{2.6}$$

if we solve equation (2.6), we obtain the following control:

$$u = -R^{-1}B^T\lambda \tag{2.7}$$

From (2.1) and (2.7), we have the following canonical differential equations that form a linear system (or Hamiltonian system) written as:

$$\begin{aligned} \dot{x} &= \frac{\partial H}{\partial \lambda} \\ &= Ax - BR^{-1}B^T\lambda, \quad x(0) = x_0 \\ \dot{\lambda} &= -\frac{\partial H}{\partial x} \\ &= -Qx - A^T\lambda \end{aligned} \tag{2.8}$$

Since the terminal cost is not defined, then there is no constraint on the final value of λ.
This is a coupled system, linear in x and λ, of order $2n \times 2n$.
These control equations can be written in matrix form as:

$$\begin{pmatrix} \dot{x} \\ \dot{\lambda} \end{pmatrix} = \begin{pmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} \tag{2.9}$$

It is not easy to solve the system described in equation (2.9), so we guess the solution of this system or the relation between x and λ in the form:

$$\lambda = Px \tag{2.10}$$

where $P \in \Re^{n \times n}$.
We introduce now an important differential equation in the linear quadratic regulator problem that is called *Matrix Riccati Equation* (MRE).
To derive this equation, we start from equation (2.10) and use (2.8) in the following way:

$$\begin{aligned} \lambda &= Px \\ \dot{\lambda} &= \dot{P}x + P\dot{X} \\ -Qx - A^T\lambda &= \dot{P}x + P(Ax - BR^{-1}B^T\lambda) \\ -Qx - A^TPx &= \dot{P}x + PAx - PBR^{-1}B^TPx \\ \dot{P}x + PAx + A^TPx - PBR^{-1}B^TPx + Qx = 0 \end{aligned}$$

From the final step, we obtain the MRE written as:

$$\dot{P} = -PA - A^TP + PBR^{-1}B^TP - Q \tag{2.11}$$

Since we have an infinite time horizon, there is no information about the terminal cost and hence λ has no constraint. In this case the steady state solution P of a so called Algebraic Riccati Equation (ARE) can be used instead of $P(t)$ (Murray, 2009).
In case when the time approaches infinty, we have:

$$\lim_{t \longrightarrow \infty} \dot{P} = 0$$

By using the limit above, we get another differential equation called *Algebraic Riccati Equation* (ARE), written as:

$$PA + A^T P - PBR^{-1}B^T P + Q = 0 \tag{2.12}$$

where P is the unique positive-definite solution.
We want now to find a state feedback control u that can be used to move any state x to the origin, so we let the system evolve in a closed-loop (Murray, 2009; Ghoreishi, Nekoui, & Basiri, 2011).
If we find the solution P of the ARE (2.12), then the optimal control u that can be used to minimize the quadratic cost function J is written as:

$$u = -R^{-1}B^T Px \tag{2.13}$$

By substituting equation (2.13) into the original system described by equation (2.1), we get the following equation:

$$\dot{x} = (A - R^{-1}B^T P)x \tag{2.14}$$

Since the matrix $A - BK$ is stable, we have closed-loop poles formed by the eigenvalues of this matrix (Ghoreishi et al., 2011).
If we solve equation (2.14) and find the optimal solution x, then we can find our optimal control u that can be used to find a minimum value of the quadratic cost function J described in equation (2.2).
We can summarize the *LQR* method as follows:

1. We start with the linear dynamical system:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \\ x(0) &= x_0 \end{aligned}$$

2. We assume that this system is controllable.

3. We define the quadratic cost function as:

$$J = \frac{1}{2}\int_0^\infty (x^T Qx + u^T Ru)dt$$

4. We choose $Q = Q^T \geq 0$ such that $Q = C^T C$ and $R = R^T > 0$

5. We find the constant solution P of the ARE:

$$PA + A^T P - PBR^{-1}B^T P + Q = 0$$

6. We find the optimal control u such that:

$$u = -R^{-1}B^T Px$$

7. We write the original system into the form:

$$\dot{x} = (A - R^{-1}B^T P)x$$

3 Requlator problem : model order reduction of type(1)

In this section we introduce the linear quadratic regulator problem for the reduced order model of a dynamical system (Kokotovic, 1976).
Our goal is to find an optimal control for the reduced system using the singular perturbation a pproximation.
Consider the linear time-invariant dynamical system defined as:

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} &= \begin{pmatrix} A_{11} & A_{12} \\ \frac{1}{\varepsilon}A_{21} & \frac{1}{\varepsilon}A_{22} \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} + \begin{pmatrix} B_1 \\ \frac{1}{\varepsilon}B_2 \end{pmatrix} u \\ y &= \begin{pmatrix} C_1 & C_2 \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} \end{aligned} \tag{3.1}$$

This system can be written in another form as :

$$\begin{aligned} \dot{x} &= A_{11}x + A_{12}z + B_1 u \\ \varepsilon\dot{z} &= A_{21}x + A_{22}z + B_2 u \end{aligned} \tag{3.2}$$

From (Daraghmeh, 2016), we see that this system can be optimized according to the following quadratic cost function:

$$J = \frac{1}{2}\int_0^\infty (y^T y + u^T Ru)dt \tag{3.3}$$

or

$$J = \frac{1}{2}\int_0^\infty (x^T Qx + u^T Ru)dt \tag{3.4}$$

where $Q = C^T C \geq 0$ and $R > 0$.
The optimal control u is defined as:

$$u = -R^{-1}\begin{pmatrix} B_1^T & \frac{1}{\varepsilon}B_2^T \end{pmatrix} P \begin{pmatrix} x \\ z \end{pmatrix} \tag{3.5}$$

where P is the solution of the Algebraic Riccati Equation (ARE):

$$PA + A^T P - PBR^{-1}B^T P + Q = 0 \tag{3.6}$$

The goal now is to solve the ARE and set $\varepsilon = 0$ to obtain a reduced equation for the ARE.
If we substitute the matrices A,B,C and Q in equation (3.6), then we have the following new form of ARE:

$$\begin{aligned} &P\begin{pmatrix} A_{11} & A_{12} \\ \frac{1}{\varepsilon}A_{21} & \frac{1}{\varepsilon}A_{22} \end{pmatrix} + \begin{pmatrix} A_{11}^T & \frac{1}{\varepsilon}A_{21}^T \\ A_{12}^T & \frac{1}{\varepsilon}A_{22}^T \end{pmatrix} P \\ &- P\begin{pmatrix} B_1 \\ \frac{1}{\varepsilon}B_2 \end{pmatrix} R^{-1}\begin{pmatrix} B_1^T & \frac{1}{\varepsilon}B_2^T \end{pmatrix} P + \begin{pmatrix} C_1^T \\ C_2^T \end{pmatrix}\begin{pmatrix} C_1 & C_2 \end{pmatrix} = 0 \end{aligned} \tag{3.7}$$

A solution of equation (3.7) can be choosen as:

$$P = \begin{pmatrix} P_{11} & \varepsilon P_{12} \\ \varepsilon P_{12}^T & \varepsilon P_{22} \end{pmatrix} \tag{3.8}$$

so we can avoid the unboundness when we set $\varepsilon \longrightarrow 0$ (Kokotovic, 1976). Substituting equation (3.8) into equation (3.7), we get :

$$\begin{pmatrix} P_{11} & \varepsilon P_{12} \\ \varepsilon P_{12}^T & \varepsilon P_{22} \end{pmatrix}\begin{pmatrix} A_{11} & A_{12} \\ \frac{1}{\varepsilon}A_{21} & \frac{1}{\varepsilon}A_{22} \end{pmatrix} + \begin{pmatrix} A_{11}^T & \frac{1}{\varepsilon}A_{21}^T \\ A_{12}^T & \frac{1}{\varepsilon}A_{22}^T \end{pmatrix}\begin{pmatrix} P_{11} & \varepsilon P_{12} \\ \varepsilon P_{12}^T & \varepsilon P_{22} \end{pmatrix}$$
$$- \begin{pmatrix} P_{11} & \varepsilon P_{12} \\ \varepsilon P_{12}^T & \varepsilon P_{22} \end{pmatrix}\begin{pmatrix} B_1 \\ \frac{1}{\varepsilon}B_2 \end{pmatrix} R^{-1} \begin{pmatrix} B_1^T & \frac{1}{\varepsilon}B_2^T \end{pmatrix}\begin{pmatrix} P_{11} & \varepsilon P_{12} \\ \varepsilon P_{12}^T & \varepsilon P_{22} \end{pmatrix}$$
$$+ \begin{pmatrix} C_1^T \\ C_2^T \end{pmatrix}\begin{pmatrix} C_1 & C_2 \end{pmatrix} = 0 \tag{3.9}$$

From equation (3.9), we get the following $(n+m)\times(n+m)$ equations:

$$\begin{aligned} 0 = {} & P_{11}A_{11} + P_{12}A_{21} + A_{11}^T P_{11} + A_{21}^T P_{12}^T \\ & - (P_{11}B_1 + P_{12}B_2)R^{-1}(B_1^T P_{11} + B_2^T P_{12}^T) \\ & + C_1^T C_1 \end{aligned} \tag{3.10}$$

$$\begin{aligned} 0 = {} & P_{11}A_{12} + P_{12}A_{22} + \varepsilon A_{11}^T P_{12} + A_{21}^T P_{22} \\ & - (P_{11}B_1 + P_{12}B_2)R^{-1}(\varepsilon B_1^T P_{12} + B_2^T P_{22}) \\ & + C_1^T C_2 \end{aligned} \tag{3.11}$$

$$\begin{aligned} 0 = {} & \varepsilon P_{12}^T A_{11} + P_{22}A_{21} + A_{12}^T P_{11} + A_{22}^T P_{12}^T \\ & - (\varepsilon P_{12}^T B_1 + P_{22}B_2)R^{-1}(B_1^T P_{11} + B_2^T P_{12}^T) \\ & + C_2^T C_1 \end{aligned} \tag{3.12}$$

$$\begin{aligned} 0 = {} & \varepsilon P_{12}^T A_{12} + P_{22}A_{22} + \varepsilon A_{12}^T P_{12} + A_{22}^T P_{22} \\ & - (\varepsilon P_{12}^T B_1 + P_{22}B_2)R^{-1}(\varepsilon B_1^T P_{12} + B_2^T P_{22}) \\ & + C_2^T C_2 \end{aligned} \tag{3.13}$$

When we set $\varepsilon = 0$ in equations (3.10)-(3.13) we obtain the following $m \times m$ reduced equation for $\bar{P}_{22}$ and written as:

$$\bar{P}_{22}A_{22} + A_{22}^T \bar{P}_{22} - \bar{P}_{22} W \bar{P}_{22} + C_2^T C_2 = 0 \tag{3.14}$$

where $W = B_2 R^{-1} B_2^T$.
Another $n \times n$ equation for $\bar{P}_{11}$ is obtained when we express $\bar{P}_{12}$ in terms of $\bar{P}_{11}$ and $\bar{P}_{22}$ and this equation takes the form:

$$\bar{P}_{11}\hat{A} + \hat{A}^T \bar{P}_{11}^T - \bar{P}_{11}\hat{B}R^{-1}\hat{B}^T \bar{P}_{11} + \hat{C}^T \hat{C} = 0 \tag{3.15}$$

where $\hat{A}$,$\hat{B}$ and $\hat{C}$ are defined in (Kokotovic & Yackel, 1972). If $(\hat{A},\hat{B})$ is controllable pair and $(\hat{A},\hat{C})$ is observable pair, then applying the implicit function theorem to equation (3.7) with equation (3.8) (Kokotovic, 1976, 1984), we have:

$$P_{ij} = \bar{P}_{ij} + O(\varepsilon), \quad i,j = 1,2 \tag{3.16}$$

If we use $\bar{P}_{ij}$ instead of P_{ij} in equation (3.16), then the feedback control in equation (3.5) becomes:

$$\begin{aligned} u &= -R^{-1} \begin{pmatrix} B_1^T & \frac{1}{\varepsilon}B_2^T \end{pmatrix} \begin{pmatrix} \bar{P}_{11} & \varepsilon\bar{P}_{12} \\ \varepsilon\bar{P}_{12}^T & \varepsilon\bar{P}_{22} \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} \\ &= -R^{-1}(B_1^T \bar{P}_{11} + B_2^T \bar{P}_{12})x - R^{-1}(\varepsilon B_1^T \bar{P}_{12} + B_2^T \bar{P}_{22})z \end{aligned} \tag{3.17}$$

From equation (3.17), the original system described by equation (3.1) becomes:

$$\begin{aligned} \dot{x} &= \left(A_{11} - B_1 R^{-1}(B_1^T \bar{P}_{11} + B_2^T \bar{P}_{12})\right)x + \left(A_{12} - B_1 R^{-1}(\varepsilon B_1^T \bar{P}_{12} + B_2^T \bar{P}_{22})\right)z \\ \varepsilon\dot{z} &= \left(A_{21} - B_2 R^{-1}(B_1^T \bar{P}_{11} + B_2^T \bar{P}_{12})\right)x + \left(A_{22} - B_2 R^{-1}(\varepsilon B_1^T \bar{P}_{12} + B_2^T \bar{P}_{22})\right)z \end{aligned} \tag{3.18}$$

If this system is asymptotically stable then from equation (3.16), we have a solution $x(t)$ and $z(t)$ with $O(\varepsilon)$ of the optimal solution (Kokotovic et al., 1976). If we assume that A_{22} is stable, then we can apply this assumption to the feedback system in equation (3.18).
If we reduce the full system in equation (3.2) using the singular approximation approximation, we obtain the following reduced order model:

$$\begin{aligned} \dot{x}_r &= A_r x_r + B_r u_r \\ y_r &= C_r x_r + D_r u_r \end{aligned} \tag{3.19}$$

where

$$A_r = A_{11} - A_{12}A_{22}^{-1}A_{21}$$
$$B_r = B_1 - A_{12}A_{22}^{-1}B_2$$
$$C_r = C_1 - C_2A_{22}^{-1}A_{21}$$
$$D_r = -C_2A_{22}^{-1}B_2$$

We define the cost quadratic function of this reduced order system as:

$$J_r = \frac{1}{2}\int_0^{\infty}(y_r^T y_r + u_r^T R_r u_r)dt \tag{3.20}$$

or, equivalently

$$J_r = \frac{1}{2}\int_0^{\infty}(x_r^T Q_r x_r + 2x_r^T C_r D_r u_r + u_r^T R_r u_r)dt \tag{3.21}$$

where $Q_r = C_r^T C_r$ and $R_r = R + D_r^T D_r$.
The optimal control for this reduced system defined as:

$$u_r = -R_r^{-1}B_r^T P_r x_r \tag{3.22}$$

where P_r is the constant solution of the following *Algebraic Riccati Equation* for the reduced system described by equations (3.19) given as:

$$P_r(A_r - B_rR_r^{-1}D_r^TC_r) + (A_r - B_rR_r^{-1}D_r^TC_r)^T P_r - P_rB_rR_r^{-1}B_r^TP_r + C_r^T(I + D_rR_rD_r^T)^{-1}C_r = 0 \tag{3.23}$$

We introduce now the following theorem that describes the relationship between the reduced Riccati Equation system (3.14)–(3.15) for the full system (3.2) after putting $\varepsilon = 0$ and the Riccati Equation (3.23) for the reduced system in equation (3.19) when we set $\varepsilon = 0$

Theorem 3.1. *If equation (3.16) holds and A_{22}^{-1} exists, then the solution P_r of equation (3.23) is identical to the solution $\bar{P}_{11}$ of equation (3.15).*

For more details, see (Kokotovic et al., 1976; Kokotovic, 1984).
According to theorem (3.1) and if we substitute the feedback optimal control u_r described by equation (3.22) into the reduced system equation (3.19), then we obtain the following system:

$$\dot{x}_r = (A_r - B_r R^{-1} B_r^T P_r) x_r \tag{3.24}$$

where $(A_r - B_r R^{-1} B_r^T P_r)$ is stable and the pair (A_r, B_r) is controllable.
If we find the optimal solution x_r of (3.24) and substitute the value into equation (3.22), then we find the optimal control for the reduced order model.

4 Requlator problem : model order reduction of type(2)

In this section, we introduce a linear dynamical continuous system with input matrix B that does not depends on ε. We want to find the optimal control for this dynamical system and then use the singular perturbation approximation to reduce this system and find the optimal control for the reduced order model.
Let us consider the following linear dynamical continuous system defined as:

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} &= \begin{pmatrix} A_{11} & A_{12} \\ \frac{A_{21}}{\varepsilon} & \frac{A_{22}}{\varepsilon} \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} u \\ y &= \begin{pmatrix} C_1 & C_2 \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} \end{aligned} \tag{4.1}$$

Another representation of the above system could be written as:

$$\begin{aligned} \dot{x} &= A_{11}x + A_{12}z + B_1 u \\ \varepsilon \dot{z} &= A_{21}x + A_{22}z + \varepsilon B_2 u \end{aligned} \tag{4.2}$$

If we assume that A_{22} is stable and A_{22}^{-1} exists, then we set $\varepsilon = 0$ to obtain the following equation:

$$\bar{z} = A_{22}^{-1} A_{21} \bar{x} \tag{4.3}$$

When we substitute equation (4.3) into equation (4.2), we get the following reduced order model:

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}\bar{u}$$
$$\bar{y} = \bar{C}\bar{x} \tag{4.4}$$

where

$$\bar{A} = A_{11} - A_{12}A_{22}^{-1}A_{21}$$
$$\bar{B} = B_1$$
$$\bar{C} = C_1 - C_2A_{22}^{-1}A_{21}$$

Our goal now is to find the optimal control for the system in equation (4.1) that minimizes the quadratic cost function J defined by the following equations:

$$J = \frac{1}{2}\int_0^\infty (y^T y + u^T Ru)dt \tag{4.5}$$

or equivalently

$$J = \frac{1}{2}\int_0^\infty (x^T Qx + u^T Ru)dt \tag{4.6}$$

where $Q = C^T C \geq 0$ and $R > 0$.
The feedback optimal control u for the original system is defined as:

$$u = -R^{-1}\begin{pmatrix} B_1^T & B_2^T \end{pmatrix} P \begin{pmatrix} x \\ z \end{pmatrix} \tag{4.7}$$

where P is the solution of the Algebraic Differential Equation defined below:

$$\begin{aligned} &P\begin{pmatrix} A_{11} & A_{12} \\ \frac{1}{\varepsilon}A_{21} & \frac{1}{\varepsilon}A_{22} \end{pmatrix} + \begin{pmatrix} A_{11}^T & \frac{1}{\varepsilon}A_{21}^T \\ A_{12}^T & \frac{1}{\varepsilon}A_{22}^T \end{pmatrix} P \\ &- P\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} R^{-1}\begin{pmatrix} B_1^T & B_2^T \end{pmatrix} P + \begin{pmatrix} C_1^T \\ C_2^T \end{pmatrix}\begin{pmatrix} C_1 & C_2 \end{pmatrix} = 0 \end{aligned} \tag{4.8}$$

We choose the solution of equation (4.8) as:

$$P = \begin{pmatrix} P_{11} & \varepsilon P_{12} \\ \varepsilon P_{12}^T & \varepsilon P_{22} \end{pmatrix} \tag{4.9}$$

to avoid the unboundness for $\varepsilon = 0$.
Equation (4.9) together with equation (4.8) give the following equation:

$$\begin{aligned}
&\begin{pmatrix} P_{11} & \varepsilon P_{12} \\ \varepsilon P_{12}^T & \varepsilon P_{22} \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ \frac{1}{\varepsilon} A_{21} & \frac{1}{\varepsilon} A_{22} \end{pmatrix} + \begin{pmatrix} A_{11}^T & \frac{1}{\varepsilon} A_{21}^T \\ A_{12}^T & \frac{1}{\varepsilon} A_{22}^T \end{pmatrix} \begin{pmatrix} P_{11} & \varepsilon P_{12} \\ \varepsilon P_{12}^T & \varepsilon P_{22} \end{pmatrix} \\
&- \begin{pmatrix} P_{11} & \varepsilon P_{12} \\ \varepsilon P_{12}^T & \varepsilon P_{22} \end{pmatrix} \begin{pmatrix} B_1 \\ \frac{1}{\varepsilon} B_2 \end{pmatrix} R^{-1} \begin{pmatrix} B_1^T & \frac{1}{\varepsilon} B_2^T \end{pmatrix} \begin{pmatrix} P_{11} & \varepsilon P_{12} \\ \varepsilon P_{12}^T & \varepsilon P_{22} \end{pmatrix} \\
&+ \begin{pmatrix} C_1^T \\ C_2^T \end{pmatrix} \begin{pmatrix} C_1 & C_2 \end{pmatrix} = 0
\end{aligned} \tag{4.10}$$

Form equation (4.10), we obtain the following set of equations:

$$\begin{aligned}
0 = {} & P_{11}A_{11} + P_{12}A_{21} + A_{11}^T P_{11} + A_{21}^T P_{12}^T \\
& - (P_{11}B_1 + \varepsilon P_{12}B_2) R^{-1} (B_1^T P_{11} + \varepsilon B_2^T P_{12}^T) \\
& + C_1^T C_1
\end{aligned} \tag{4.11}$$

$$\begin{aligned}
0 = {} & P_{11}A_{12} + P_{12}A_{22} + \varepsilon A_{11}^T P_{12} + A_{21}^T P_{22} - (P_{11}B_1 \\
& + \varepsilon P_{12}B_2) R^{-1} (\varepsilon B_1^T P_{12} + \varepsilon B_2^T P_{22}) \\
& + C_1^T C_2
\end{aligned} \tag{4.12}$$

$$\begin{aligned}
0 = {} & \varepsilon P_{12}^T A_{11} + P_{22}A_{21} + A_{12}^T P_{11} + A_{22}^T P_{12}^T \\
& - (\varepsilon P_{12}^T B_1 + \varepsilon P_{22}B_2) R^{-1} (\varepsilon B_1^T P_{11} + \varepsilon B_2^T P_{12}^T) \\
& + C_2^T C_1
\end{aligned} \tag{4.13}$$

$$\begin{aligned}
0 = {} & \varepsilon P_{12}^T A_{12} + P_{22}A_{22} + \varepsilon A_{12}^T P_{12} + A_{22}^T P_{22} \\
& - (\varepsilon P_{12}^T B_1 + \varepsilon P_{22}B_2) R^{-1} (\varepsilon B_1^T P_{12} + \varepsilon B_2^T P_{22}) \\
& + C_2^T C_2
\end{aligned} \tag{4.14}$$

When we set $\varepsilon = 0$ in equations (4.11)-(4.14) we obtain the following reduced Riccati equations:

$$\bar{P}_{11}A_{11}+\bar{P}_{12}A_{21}+A_{11}^T\bar{P}_{11}^T+A_{21}^T\bar{P}_{12}^T-\bar{P}_{11}B_1R^{-1}B_1^T\bar{P}_{11}+C_1^TC_1=0 \quad (4.15)$$

$$\bar{P}_{11}A_{12}+\bar{P}_{12}A_{22}+A_{21}^T\bar{P}_{22}+C_1^TC_2=0 \quad (4.16)$$

$$\bar{P}_{22}A_{21}+A_{12}^T\bar{P}_{11}+A_{22}^T\bar{P}_{12}^T+C_2^TC_1=0 \quad (4.17)$$

$$\bar{P}_{22}A_{22}+A_{22}^T\bar{P}_{22}+C_2^TC_2=0 \quad (4.18)$$

We write $\bar{P}_{12}$ and $\bar{P}_{12}^T$ in equations (4.16),(4.17) in terms of $\bar{P}_{11}$ and $\bar{P}_{22}$ as follows:

$$\bar{P}_{12}=-(\bar{P}_{11}A_{12}+A_{21}^T\bar{P}_{22}+C_1^TC_2)A_{22}^{-1} \quad (4.19)$$

$$\bar{P}_{12}^T=-(A_{22}^T)^{-1}(\bar{P}_{22}A_{21}+A_{12}^T\bar{P}_{11}+C_2^TC_1) \quad (4.20)$$

Equation (4.18) can be expressed in different form as:

$$A_{21}^T(A_{22}^T)^{-1}\bar{P}_{22}A_{21}+A_{21}^T\bar{P}_{22}A_{22}^{-1}A_{21}=-A_{21}^T(A_{22}^T)^{-1}C_2^TC_2A_{22}^{-1}A_{21} \quad (4.21)$$

Substituting equations (4.19) and (4.20) into equation (4.15) and using equation (4.21) we obtain:

$$\bar{P}_{11}\hat{A}+\hat{A}^T\bar{P}_{11}-\bar{P}_{11}\hat{B}R^{-1}\hat{B}^T\bar{P}_{11}+\hat{C}^T\hat{C}=0 \quad (4.22)$$

where

$$\begin{aligned}\hat{A}&=A_{11}-A_{12}A_{22}^{-1}A_{21}\\ \hat{B}&=B_1\\ \hat{C}&=C_1-C_2A_{22}^{-1}A_{21}\end{aligned} \quad (4.23)$$

If we assume the pair $(\hat{A},\hat{B})$ is controllable, then the values of P_{ij} and $\bar{P}_{ij},\ i,j=1,2$ satisfy equation (3.16).

The feedback optimal control defined in equation (4.7) together with the result in equation (3.16) can be written as:

$$\begin{aligned}u&=-R^{-1}\begin{pmatrix} B_1^T & B_2^T\end{pmatrix}\begin{pmatrix}\bar{P}_{11} & \varepsilon\bar{P}_{12}\\ \varepsilon\bar{P}_{12}^T & \varepsilon\bar{P}_{22}\end{pmatrix}\begin{pmatrix}x\\ z\end{pmatrix}\\ &=-R^{-1}(B_1^T\bar{P}_{11}+\varepsilon B_2^T\bar{P}_{12})x-R^{-1}(\varepsilon B_1^T\bar{P}_{12}+\varepsilon B_2^T\bar{P}_{22})z\end{aligned} \quad (4.24)$$

We can use the result found in equation (4.24) to write a new representation of the original system described by equation (4.2) as:

$$\begin{aligned}
\dot{x} &= \left(A_{11} - B_1R^{-1}(B_1^T\bar{P}_{11} + \varepsilon B_2^T\bar{P}_{12})\right)x + \left(A_{12} - B_1R^{-1}(\varepsilon B_1^T\bar{P}_{12} + \varepsilon B_2^T\bar{P}_{22})\right)z \\
\varepsilon\dot{z} &= \left(A_{21} - \varepsilon B_2R^{-1}(B_1^T\bar{P}_{11} + \varepsilon B_2^T\bar{P}_{12})\right)x + \left(A_{22} - \varepsilon B_2R^{-1}(\varepsilon B_1^T\bar{P}_{12} + \varepsilon B_2^T\bar{P}_{22}\right)z
\end{aligned} \tag{4.25}$$

If the system in equation (4.25) is asymptotically stable and if equation (3.16) holds, then we can compute the solution $x(t)$ and $z(t)$ within the $O(\varepsilon)$ of the optimal control.
The next step now is to find a feedback optimal control for the reduced system defined in equation (4.4) that can be used to minimizes the quadratic cost function $\bar{J}$ defined as:

$$\bar{J} = \frac{1}{2}\int_0^\infty (\bar{y}^T\bar{y} + \bar{u}T\bar{R}\bar{u})dt \tag{4.26}$$

or equivalently

$$\bar{J} = \frac{1}{2}\int_0^\infty (\bar{x}^T\bar{Q}\bar{x} + \bar{u}^T\bar{R}\bar{u})dt \tag{4.27}$$

where $\bar{Q} = \bar{C}^T\bar{Q} \geq 0$ and $\bar{R} = R > 0$.
We define the optimal control for the reduced system (4.4) as:

$$\bar{u} = -\bar{R}^{-1}\bar{B}^T\bar{P}\bar{x} \tag{4.28}$$

where $\bar{P}$ is the solution of the following Algebraic Riccati Equation for the reduced system in equation (4.4), defined as:

$$\bar{P}\bar{A} + \bar{A}^T\bar{P} - \bar{P}\bar{B}\bar{R}^{-1}\bar{B}^T\bar{P} + \bar{C}^T\bar{C} = 0 \tag{4.29}$$

Since A_{22} is stable and A_{22}^{-1} is exists, then the solution of equation (4.29) is the same as the solution of equation (4.22), thus we have:

$$\bar{P} = \bar{P}_{11} \tag{4.30}$$

By usig the feedback optimal control in equation (4.28) and the solution $\bar{P}$ in equation (4.29), then we obtain the following reduced system derived from the reduced system in equation (4.4) that has the form:

$$\begin{aligned} \dot{\bar{x}} &= \left(\bar{A} - \bar{B}\bar{R}^{-1}\bar{B}^T\bar{P}\right)\bar{x} \\ \bar{y} &= \bar{C}\bar{x} \end{aligned} \tag{4.31}$$

where

$$\begin{aligned} \bar{A} &= A_{11} - A_{12}A_{22}^{-1}A_{21} \\ \bar{B} &= B_1 \\ \bar{C} &= C_1 - C_2A_{22}^{-1}A_{21} \end{aligned}$$

We assume that the matrix $\bar{A} - \bar{B}\bar{R}^{-1}\bar{B}^T\bar{P}$ is stable and the pairs $(\bar{A}, \bar{B})$, $(\bar{A}, \bar{C})$ are controllable and observable respectively.
By solving the reduced system in equation (4.31), the solution $\bar{x}(t)$ is used to find the feedback control $\bar{u}$ which is important to find the minimum value of the quadratic cost function $\bar{J}$.

5 Numerical Example : Building model

In this section we introduce a numerical example that describe the behavior of the dynamical system. As an application, we consider the Building model system from the SLICOT library (Chahlaoui & Van Dooren, 2005; Steinbuch, Van Groos, Schootstra, Wortelboer, & Bosgra, 1998), which is a model of a building (the Los Angeles University Hospital) with $N = 48$ degrees of freedom. We start by applying different types of singular perturbation approximation method to the system with homogeneous initial conditions $x = 0$.

To find an optimal control U_1 for the original system and u for the reduced order system, we apply different types of the singular perturbation approximation methods to CD-player system example. The size of the system is $N = 48$ and the size of the reduced model is $r = 2$. The optimal control is computed by using the results in sections (3) and (4). We use the solution of the Riccati equation P of the original system and use it to

find the value of U_1. To find the solution P_r of the Riccati equation for the reduced system, we apply the approaches in sections (3) and (4). Since the first block P_{11} of P is equal to the value of P_r, so we can extended P using P_r as the first block and the rest blocks are zero to obtain a new solution of the Riccati equation denoted by $\tilde{P}_{11}$.
We can use the value of $\tilde{P}_{11}$ to find another optimal control for the full system denoted by U_2 and hence we compute the $\|U_1 - U_2\|_{L_2}$ norm. Figure (1) represents the plots of the two optimal controls U_1, U_2 and $(U_1 - U_2)$ by applying different types of the singular approximation perturbation to the full and reduced systems indicate in (3) and (4) .
Finally, Table (5.1) contains the values of $\|U_1 - U_2\|_{L_2}$ and $\|P_{11} - \tilde{P}_{11}\|_{L_2}$

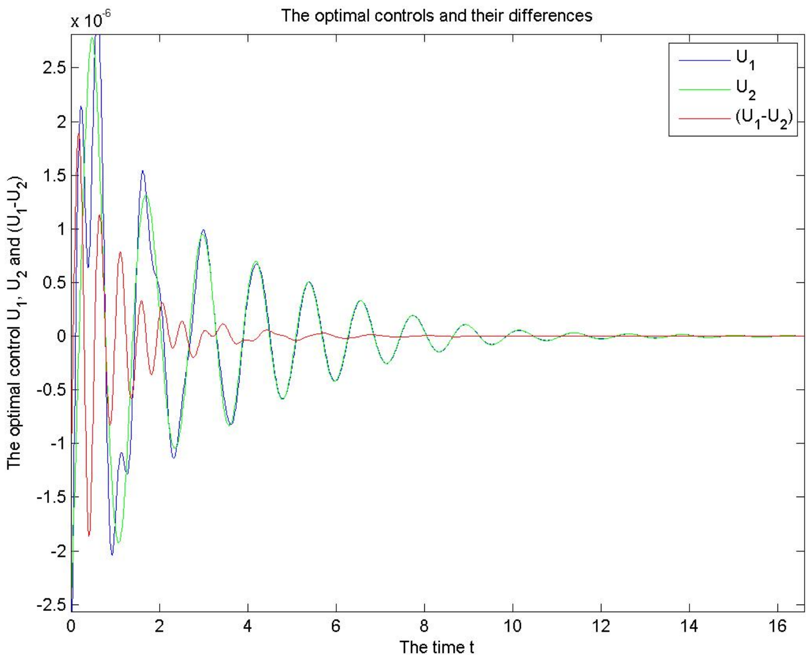

Figure 1: The optimal controls of the building model system

by applying the balanced truncation and singular perturbation approximation to the building model system.

Table (5.1): The L^2 norm of $(U_1 - U_2)$ and $(P_{11} - \tilde{P}_{11})$ of the building model.

r_s	$\|U_1 - U_2\|_{L_2}$ SPA1	$\|P_{11} - \tilde{P}_{11}\|_{L_2}$	$\|U_1 - U_2\|_{L_2}$ SPA2	$\|P_{11} - \tilde{P}_{11}\|_{L_2}$
2	8.0741×10^{-15}	$3.4098\times10^{-}9$	8.0651×10^{-15}	3.3189×10^{-9}
4	1.1778×10^{-15}	$3.2719\times10^{-}9$	1.1665×10^{-15}	3.1688×10^{-9}
6	3.6256×10^{-16}	3.7787×10^{-9}	3.5344×10^{-16}	3.7899×10^{-9}
8	8.7808×10^{-17}	1.3664×10^{-9}	8.6817×10^{-17}	1.2345×10^{-9}
10	9.0881×10^{-17}	1.2910×10^{-9}	9.0779×10^{-17}	1.1889×10^{-9}
12	1.6477×10^{-18}	1.1948×10^{-9}	1.4576×10^{-18}	6.8224×10^{-9}

6 Conclusion

In this article the LQR has been used to find an optimal control that minimizes the constraints on the states and the control input. This has been carried out by using formal asymptotes for the Pontryagin maximum principle (PMP) and the Hamiltonian function together with the Riccati equation.
We have extended the model reduction using singular perturbation approximation method of the balanced infinite dimensional systems. The reduced-order model was obtained by setting the derivative of all states corresponding to the smaller Hankel singular values of the balanced systems equal to zero.
The optimal control of the full-order systems and of reduced-order systems by the singular perturbation approximation are shown in Figure (1). The optimal errors of both model reduction methods are given in table (5.1), respectively under the L_2-norm with zero initial condition. Numerical results show clearly that the singular perturbation give good closed-loop performance.

References

Chahlaoui, Y., & Van Dooren, P. (2005). Benchmark examples for model reduction of linear time invariant dynamical systems. In P. Benner, D. C. Sorensen, & V. Mehrmann (Eds.), *Dimension reduction of large-scale systems* (Vol. 45, p. 379-392). Springer, Berlin.

Daraghmeh, A. (2016). *Model order reduction of linear control systems: Comparison of balanced truncation and singular perturbation approximation with application to optimal control.* PhD Thesis, Berlin Freie Universität,Germany.

F.L.Lewis, & V.L.Syrmos. (1995). *Optimal control, 2^{nd} edition.* John Wiley & Sons,Inc.

Ghoreishi, S. A., Nekoui, M. A., & Basiri, S. O. (2011). Optimal design of lqr weighting matrices based on intelligent optimization methods. *International Journal of Intelligent Information Processing*, *2*(1).

G.P.Liu, & R.J.Patton. (1998). *Eigenstructure assignment for control system design.* John Wiley & Sons,Inc.

J.W.Choi, & Y.B.Seo. (1998). *"control design methology using ealqr,".* in Proceedings of the IEEE 37th SICE Annual Conference.

Knowles, G. E. (1981). *An introduction to applied optimal control.* Academic Press.

Kokotovic, P. V. (1976). Singular perturbations in optimal control. *JOURNAL OF MATHEMATICS*, *6*(4).

Kokotovic, P. V. (1984). Applications of singular perturbation techniques to control problems. *SIAM review*, *26*(4), 501–550.

Kokotovic, P. V., O'malley, R. E., & Sannuti, P. (1976). Singular perturbations and order reduction in control theory—an overview. *Automatica*, *12*(2), 123–132.

Kokotovic, P. V., & Yackel, R. A. (1972). Singular perturbation of linear regulators: basic theorems. *Automatic Control, IEEE Transactions on*, *17*(1), 29–37.

Murray, R. M. (2009). Optimization-based control. *California Institute of Technology, CA*.

O'Malley, R., Jr. (1972). The singularly perturbed linear state regulator problem. *SIAM Journal on Control*, *10*(3), 399–413.

O'Malley Jr, R. E. (1975). On two methods of solution for a singularly perturbed linear state regulator problem. *SIAM Review*, *17*(1), 16–

37.

Steinbuch, M., Van Groos, P. J., Schootstra, G., Wortelboer, P. M., & Bosgra, O. H. (1998). μ-synthesis for a compact disc player. *International Journal of Robust and Nonlinear Control*, *8*(2), 169–189.