On The Numerical Simulation of Heat Conduction Problem

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Abstract

In this paper we focus on the numerical simulation of the steady state heat conduction in a three dimensional enclosure geometry without the presence of heat source. This physical phenomena is governed by a boundary integral equation of the second kind. For the discretization of the boundary integral equation we have used the boundary element method based on the Galerkin weighted residuals method. The system of linear equation which subsequently arise has been solved by the conjugate gradient method. To demonstrate the high efficiency of this method a numerical experiment has been constructed.

Keywords: Heat conduction, Fredholm integral equation, Boundary element method, Multigrid iterations, conjugate gradient scheme .

1 Introduction

The steady state heat conduction taking place in an enclosure $\Omega \in \mathbb{R}^3$ with boundary Γ (without the presence of internal heat source) can be described by the following boundary integral equation (see[6])

$$0.5T(\vec{p}) = \int_{\Gamma} [\overset{*}{T}(\vec{r},\vec{p})q(\vec{r}) - \overset{*}{q}(\vec{r},\vec{p})T(\vec{r})]d\Gamma(\vec{r}), \qquad \vec{r},\vec{p} \in \Gamma \tag{1.1}$$

with the temperature at internal points is expressed in terms of the boundary temperatures and boundary fluxes. r and p stand for the current and source points respectively. The temperature field $\overset{*}{T}(\vec{r},\vec{p})$ and the heat flux $\overset{*}{q}(\vec{r},\vec{p})$ known as kernels of the integral equation are given through

$$\overset{*}{T}(\vec{r},\vec{p}) = \frac{1}{4\pi}.\frac{1}{\left|\vec{r}-\vec{p}\right|} \tag{1.2}$$

$$\overset{*}{q}(\vec{r},\vec{p}) = \frac{1}{4\pi}.\frac{(\vec{p}-\vec{r})}{\left|\vec{p}-\vec{r}\right|} \tag{1.3}$$

In addition to some previous work [3,4,6,11,12,15,16] involving the heat radiation integral equation we are aware of some other work [1,2,5,8,11,12,14,17] on heat conduction together with other heat transfer modes. Our main concern in this work is to focus on the numerical solution of the boundary integral equation (1.1). This will be achieved by introducing the boundary element method based on the Galerkin weighted residual method that will convert

the original boundary integral equation (1.1) to a system of linear equations. This linear system will be solved iteratively using multigrid methods. In fact, multigrid methods are among the most efficient methods for solving the linear system associated with the numerical solution of the integral equation. The characteristic feature of the multigrid method is its fast convergence in comparison to other iterative methods. Numerical example is considered to demonstrate the high performance of these iterations.

2 Numerical realization of (1.1)

2.1 Construction of the system of equations

For the numerical realization of equation (1.1) we use the boundary element method based on the Galerkin weighted residuals method. The temperature T and the heat flux q within each boundary element are approximated as follows:

$$T(r) = \sum_{i=1}^{N} T_i \varphi_i(r) \tag{2.1}$$

$$q(r) = \sum_{i=1}^{N} q_i \varphi_i(r) \tag{2.2}$$

where T_i and q_i are the values of the nodal point r_i of the temperature and the heat flux respectively. φ_i are basis functions of the nodal point r_i. Inserting (2.1) and (2.2) into equation (1.1) and collocation at one nodal point p_k yields a linear equation linking nodal temperatures and heat fluxes [6],

$$\sum_{j=1}^{N} H_{kj} T_j = \sum_{j=1}^{N} G_{kj} q_j \tag{2.3}$$

where

$$H_{kj} = \sum_{j=1}^{N} \left(\int_{\Gamma} \overset{*}{q}(r, p_k) \varphi_j(r) d\Gamma(r) + 0.5 \delta_{kj} \right) \tag{2.4}$$

and

$$G_{kj} = \sum_{j=1}^{N} \left(\int_{\Gamma} \overset{*}{T}(r, p_k) \varphi_j(r) d\Gamma(r) \right) \tag{2.5}$$

Equation (2.3) can be written for a sequence of collocation points $p_k, k = 1, 2, ..., N$ yields a set of linear equations

$$HT = Gq \tag{2.6}$$

The vectors T and q contain values of temperatures and heat fluxes of collocation points. H is called the temperature influence matrix and G is called the heat flux influence matrix. The above integrals in (2.4) and (2.5) can be evaluated numerically using numerical quadrature [6,15,16]. This can be accomplished by transforming the integral over an arbitrary shaped

boundary element into an integral over a unit square. Once the transformation is carried out the value of the appropriate integral is computed using numerical quadratures. Using shape functions ψ_i, one can approximate the global coordinate of the current point r laying within a boundary element as follows :

$$x = \sum_{i=1}^{L} x_i \psi_i(\xi, \zeta) \tag{2.7}$$

$$y = \sum_{i=1}^{L} y_i \psi_i(\xi, \zeta) \tag{2.8}$$

$$z = \sum_{i=1}^{L} z_i \psi_i(\xi, \zeta) \tag{2.9}$$

where x_i, y_i and z_i are the Cartesian coordinates of the nodal points defining the geometry of the element. By virtue of (2.7)-(2.9), it can be readily shown [6] that the integrals in (2.4) and (2.5) take the following form :

$$H_{kj} = \int_{-1}^{1}\int_{-1}^{1} \overset{*}{q}\left[r(\xi,\zeta), p_k\right]\varphi_j(\xi,\zeta)\left|N_r(\xi,\zeta)\right| d\xi d\zeta + 0.5\delta_{kj} \tag{2.10}$$

and

$$G_{kj} = \int_{-1}^{1}\int_{-1}^{1} \overset{*}{T}\left[r(\xi,\zeta), p_k\right]\varphi_j(\xi,\zeta)\left|N_r(\xi,\zeta)\right| d\xi d\zeta. \tag{2.11}$$

To solve equation (2.6) the values of the temperatures and heat fluxes prescribed as

$$T(r) = \widetilde{T} \qquad \text{and} \qquad q(r) = \widetilde{q} \tag{2.12}$$

are inserted into equation (2.6). Consequently, we arrive at the set of linear equations having the form

$$Au = f \tag{2.13}$$

where the entries of matrix A are defined as

$$A = \begin{cases} H_{k_j} & \text{if at point } r_j \text{ the heat flux is known} \\ -G_{k_j} & \text{if at point } r_j \text{ the temperature is known} \end{cases} \tag{2.14}$$

the coefficients of the vector of unknowns u are

$$u = \begin{cases} q_k & \text{if at point } \ r_j \text{ the temperature is given} \\ T_k & \text{if at point } r_j \text{ the heat flux is given} \end{cases} \tag{2.15}$$

and the coefficients of the right-hand vectors are computed as

$$f_k = -\sum_{i_T} H_{ki_T}\widetilde{T}_{i_T} + \sum_{i_q} G_{ki_q}\widetilde{q}_{i_q} \tag{2.16}$$

where $\widetilde{T}_{i_T}$ and $\widetilde{q}_{i_q}$ denote the prescribed values of temperature and heat flux respectively.

2.2 Iteration schemes for (2.13)

Iterative methods for solving this problem are formulated as follows:

$$u^{(i+1)} = Mu^{(i)} + Nf \tag{2.17}$$

where M and N are constructed in such a way that given an arbitrary initial vector $u^{(0)}$, the sequences $u^{(i)}$, $i = 0, 1, ...$, converges to the solution $u = A^{-1}f$. This method is called the Picard iteration. The method (2.17) converges if and only if $\rho(M) < 1$. A sufficient convergence condition is the matrix norm estimate: $||M|| < 1$.

2.2.1 The conjugate gradient method

Since the matrix A_n is symmetric and positive definite, the conjugate gradient method (Cg-method) can be applied to solve the linear system of equations (2.17). The Cg-method is a very effective scheme for solving symmetric and positive definite systems. It is given by the following algorithm, see [15].

1. Choose an initial vector $u_n^{(0)}$ and compute:

$$r_0 = A_n u_n^{(0)} - f_n, \text{set} \quad p_0 = r_0$$

2. For $\kappa \geq 0$ compute :

$$\alpha_\kappa = \frac{r_\kappa^T p_\kappa}{p_\kappa^T A_n p_\kappa}$$

$$u_n^{(\kappa+1)} = u_n^{(\kappa)} + \alpha_\kappa p_\kappa$$

$$r_{\kappa+1} = A_n u_n^{(\kappa+1)}$$

3. Stop the calculation if :

$$\frac{||r_{\kappa+1}||_2}{||r_\kappa||_2} < \epsilon$$

4. Otherwise compute :

$$\beta = \frac{r_{\kappa+1}^T A_n p_\kappa}{p_\kappa^T A_n p_\kappa}$$

$$p_{\kappa+1} = r_{\kappa+1} + \beta_\kappa p_\kappa$$

Algorithm 2.1: Cg-method

Theorem 2.1. *[15] For the positive definite matrix A_n the conjugate gradient method converges and fulfills the following error estimate:*

$$||e^{(i)}||_{A_n} \leq 2\left(\frac{(\kappa(A_n)-1)^{\frac{1}{2}}}{(\kappa(A_n)+1)^{\frac{1}{2}}}\right)^{(i)} ||e^{(0)}||_{A_n}$$

with

$$||e^{(i)}||_{A_n} = ||u_n^{(i)} - u_n||_{A_n} \quad and \quad ||e^{(0)}||_{A_n} = ||u_n^{(0)} - u_n||_{A_n}$$

3 Numerical example and results

Since the convergence requirements for the conjugate gradient method are satisfied [13], we can now apply algorithm (2.1) to solve the linear system (2.13). For the numerical applications, we consider a cylindrical enclosure geometry whose boundary Γ has the following parametric representation:

$$\Gamma = \left\{ \vec{r} \in \mathbb{R}^3, \vec{r} = \begin{pmatrix} \cos 2\pi t \\ \sin 2\pi t \\ 4z \end{pmatrix}, (t,z) \in (0,1)^2 \right\}$$

The entries of the matrix A as well as the entries of the vector f have been computed numerically. To keep the numerical integration error small, we handle the singularity of the integral kernels by employing double partial derivatives, see [10]. Note that the step size h_k is associated with the dimension parameter n_k, where $h_k = \frac{1}{n_k}$ with $n_k = 2^k$ and k is called the level number. Table (3.1) shows the numerical results for this case. It contains both the number of iteration steps and the CPU-time in seconds required by each iteration.

	Picard Iteration		Cg-Scheme	
n_k	No. of iteration steps	second	No. of iteration steps	second
32	10	< 1	12	< 1
64	10	< 1	13	< 1
128	10	0.75	15	< 1
256	10	2.92	15	< 1
512	10	11.38	15	< 1
1024	10	44.41	15	3.05

Table 3.1: Numerical results

4 Conclusions

The numerical results for Picard iteration and Cg-Scheme shown in Table(3.1) illustrate clearly that both these schemes require less number of iteration steps and CPU-time in comparison to other iterations [1,15]. This demonstrates that one of the characteristic features of the conjugate gradient schemes is its fast convergence. The convergence speed does not deteriorate when the discretization is refined, where as other classical iterative methods slow down for decreasing grid size. As consequence one obtains an acceptable approximation of the discrete problem at the expense of the computational work proportional to the number of unknowns, which is also the number of equations of the system. It is not only complexity which is optimal, also the constant of proportionality is so small that other methods can hardly surpass the conjugate gradient efficiency.

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