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TRIVIAL EXTENSIONS SUBJECT TO SEMI-REGULARITY AND SEMI-COHERENCE

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Abstract. In this paper, we investigate the transfer of Matlis' semi-regularity and semi-coherence in trivial ring extensions issued from rings (with zero-divisors). We use the obtained results to enrich the literature with new examples of semi-regular or semi-coherent rings issued as trivial extensions and validate some questions left open in the literature.

1. Introduction

Throughout, all rings are assumed to be commutative with identity element and all modules are unital. A ring R is coherent if every finitely generated ideal of R is finitely presented (equivalently, if every finitely generated submodule of a free R-module is finitely presented). Common examples of coherent rings are Noetherian rings, valuation rings, semi-hereditary rings, and von Neumann regular rings (i.e., every module is flat). For more details on coherence, see [16, 17].

In [29], Matlis proved that the class of coherent rings is precisely the class of rings for which the endomorphism ring of any infective module is a flat module. He then, in [30], called a ring R *semi-coherent* if $\hom_R(M,N)$ is a submodule of a flat R-module for any injective R-modules M and N. Integral domains and coherent rings are semi-coherent. He also called a ring R *semi-regular* if any R-module can be embedded in a flat module (equivalently; if every injective R-module is flat). He proved that a reduced ring is semi-regular if and only if it is von Neumann regular; and a Noetherian ring is semi-regular if and only if it is self-injective (i.e., quasi-Frobenius); showing thus that von Neumann regular rings and quasi-Frobenius rings are both extreme examples of semi-regular rings. Also, he showed that a ring R is semi-regular if and only if R is coherent and R_M is semi-regular for every maximal ideal M of R.

The notion of semi-regular ring was extensively studied in the literature in commutative and non-commutative settings and was very often termed as IF

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ring (cf. [9, 10, 12, 23, 40]). Finally, recall that an R-module E is fp-injective if $\mathrm{Ext}^1_R(M,E)=0$ for every finitely presented R-module M [15, IX-3]; and R is self fp-injective if it is fp-injective over itself. Also, R is semi-regular if and only if R is self fp-injective and coherent [23, Theorem 3.10] or [9, Theorem 2].

Let A be a commutative ring and E an A-module. The trivial ring extension of A by E is the ring $R:=A\ltimes E$ with the underlying group being $A\times E$ and the multiplication is defined by $(a,e)(b,f)=(ab,af+be)$. It is also called the (Nagata) idealization of E over A and is denoted by $A(+)E$. This construction was introduced by Nagata in 1962 [32] in order to facilitate interaction between rings and their modules. The literature abounds of ideal-theoretic and ring-theoretic studies of trivial ring extensions (or idealizations); see, for instance, [1, 5, 11, 13, 18, 19, 20, 26, 27, 33, 34, 35, 36, 37, 38, 41]. For basic details on trivial extensions (or idealizations), see Glaz's and Huckaba's respective books [16, 22], and also D. D. Anderson & Winders' survey paper [4]. For homological aspects of trivial extensions and other similar commutative extensions, see for instance [1, 8, 25, 28, 31, 42].

In 2009, Kourki [26] studied properties of the trivial ring extension $R:=A\ltimes E$, including when R is a semi-Goldie ring (i.e., it does not contain a direct sum of infinitely many nonzero ideals), when R is finitely cogenerated (i.e., its socle is finitely generated and essential in it); and when R is quasi-Frobenius. Very recently, we characterized semi-regular trivial ring extensions issued from integral domains [3, Theorem 2.10]. Also, our work on Zaks' conjecture on rings with semi-regular proper homomorphic images features necessary and sufficient conditions for trivial ring extensions issued from local rings to inherit residually the notion of semi-regularity (i.e., all proper homomorphic images are semi-regular) [2].

In this paper, we investigate the transfer of the notions of Matlis' semi-regularity and semi-coherence in trivial ring extensions issued from rings (with zero-divisors). We use the obtained results to enrich the literature with new examples of semi-regular or semi-coherent rings issued as trivial ring extensions and validate some questions left open in the literature.

Throughout, for a ring A, let $Q(A)$ denote its total ring of quotients, $\mathrm{Z}(A)$ the set of its zero-divisors, and $\mathrm{Max}(A)$ the set of its maximal ideals. For an ideal I of A, $\mathrm{Ann}(I)$ will denote the annihilator of I. For the reader's convenience, Figure 1 displays a diagram of implications summarizing the relations among the main notions involved in this work.

2. Transfer of semi-regularity

This section investigates the transfer of semi-regularity in trivial ring extensions issued from (local) rings (i.e., with zero-divisors). A ring R is arithmetical if every finitely generated ideal of R is locally principal [14, 24]; and R is a chained ring

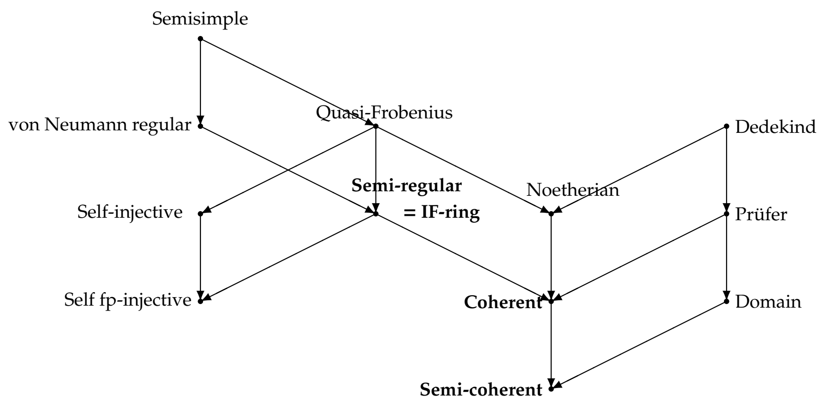

FIGURE 1. A ring-theoretic perspective for semi-regularity and semi-coherence

if R is local and arithmetical (i.e., its ideals are linearly ordered with respect to inclusion) [6, 7, 22]. Recall also that a ring is quasi-Frobenius if it is Noetherian and self-injective.

Throughout, for a ring A and an A-module E, let E_t denote the set of torsion elements of E; namely

$$E_t = \left\{e \in E \mid ae = 0 \text{ for some } 0 \neq a \in A\right\}.$$

The first result of this section establishes conditions under which some trivial extensions of local rings inherit semi-regularity. It also establishes a correlation with the notions of quasi-Frobenius ring and chained ring. Recall, for convenience, that prime (resp., maximal) ideals of a trivial extension $A \ltimes E$ have the form $p \ltimes E$, where p is a prime (resp., maximal) ideal of A [22, Theorem 25.1(3)].

Theorem 2.1. *Let $(A, \mathfrak{m})$ be a local ring, E a nonzero A-module with $\mathfrak{m}E_t = 0$ (e.g., E torsion free or $\frac{A}{\mathfrak{m}}$-vector space), and $R := A \ltimes E$. The following statements are equivalent:*

(1) *R is semi-regular;*
(2) *R is quasi-Frobenius;*
(3) *A is a chained ring, $\mathfrak{m}^2 = 0$, and $E \cong A$.*

Moreover, if any one condition holds, R is principal if and only if A is a field.

Proof. A quasi-Frobenius ring is semi-regular [30, Proposition 3.4]. So, we will prove the implications (1) $\Rightarrow$ (3) $\Rightarrow$ (2).

(1) $\Rightarrow$ (3) Assume R is semi-regular and let us envisage two cases.

Case 1: Suppose $E_t = E$. In this case, observe that $\mathfrak{m}E = 0$. We first prove, by way of contradiction, that A is a field. Deny and let $0 \neq x \in \mathfrak{m}$. Then,

$$\mathrm{Ann}_R(x, 0) = \mathrm{Ann}_A(x) \ltimes E.$$

Moreover, the facts $x \neq 0$ and $\mathrm{Ann}_A(x) \ltimes E \neq 0$ yield, respectively,

$$\mathrm{Ann}_A(x) \subseteq \mathfrak{m} \text{ and } \mathrm{Ann}_R(\mathrm{Ann}_A(x) \ltimes E) \subseteq \mathfrak{m} \ltimes E.$$

By semi-regularity of R, we obtain

$$\begin{array}{rcl} Ax \ltimes 0 & = & R(x,0) \\ & = & \mathrm{Ann}_R(\mathrm{Ann}_R(x,0)) \\ & = & \mathrm{Ann}_R(\mathrm{Ann}_A(x) \ltimes E) \\ & = & \big(\mathrm{Ann}_A(\mathrm{Ann}_A(x)) \cap \mathfrak{m}\big) \ltimes E. \end{array}$$

It follows that $E = 0$, the desired contradiction. Therefore, A is a field. Next, let e be a nonzero vector in E. Clearly, semi-regularity of R combined with the fact that e is torsion free yields

$$(*) \qquad \begin{array}{rcl} 0 \ltimes Ae & = & R(0,e) \\ & = & \mathrm{Ann}_R(\mathrm{Ann}_R(0,e)) \\ & = & \mathrm{Ann}_R(\mathrm{Ann}_A(e) \ltimes E) \\ & = & \mathrm{Ann}_R(0 \ltimes E) \\ & = & 0 \ltimes E \end{array}$$

so that $E = Ae \cong A$, as desired.

Case 2: Suppose $E_t \subsetneq E$ and let $e \in E \setminus E_t$. The same arguments used in $(*)$ yield $E = Ae \cong A$. Therefore, we may assume that $R := A \ltimes A$ with $\mathfrak{m}\mathrm{Z}(A) = 0$.

If $\mathrm{Z}(A) = 0$, then, for any $0 \neq a \in A$, $Aa = A$ again by $(*)$, hence A is a field.

If $\mathrm{Z}(A) \neq 0$, then $\mathfrak{m} \subseteq \mathrm{Z}(A)$, hence $\mathfrak{m} = \mathrm{Z}(A)$, whence $\mathfrak{m}^2 = 0$.

It remains to prove that A is a chained ring. Next, let t be a nonzero arbitrary element of $\mathfrak{m}$. Observe that, for $(x,y) \in \mathrm{Ann}_R(\mathfrak{m} \ltimes A)$, $(x,y)(0,1) = 0$ yields $x = 0$, and $(0,y)(t,0) = 0$ yields $y \in \mathfrak{m}$. So, we have

$$\begin{array}{rcl} 0 \ltimes (t) & = & \mathrm{Ann}_R(\mathrm{Ann}_R(0,t)) \\ & = & \mathrm{Ann}_R(\mathfrak{m} \ltimes A) \\ & = & 0 \ltimes \mathfrak{m} \end{array}$$

so that $\mathfrak{m} = (t)$. Now, let I be a nonzero proper ideal of A (i.e., $0 \subsetneq I \subseteq \mathfrak{m}$) and let $0 \neq a \in I$. Necessarily, $I = \mathfrak{m} = (a)$, proving that A is a chained ring (in fact, principal).

(3) $\Rightarrow$ (2) Assume A is a chained ring, $\mathfrak{m}^2 = 0$, and $E \cong A$. Observe first that the assumption $\mathfrak{m}^2 = 0$ forces $\mathfrak{m}\mathrm{Z}(A) = 0$. So, we may assume that $R := A \ltimes A$. Also, notice that the only ideals of A are (0) and $\mathfrak{m}$, and then A is necessarily principal. Hence A is Artinian. Moreover, $\mathrm{Ann}_A(\mathfrak{m}) = \mathfrak{m}$ and so the socle of A is square free by [26, Lemma 3.1]. Therefore, by Kourki's result [26, Theorem 3.6], R is quasi-Frobenius.

For the proof of the last statement of the theorem, recall first from [4, Theorem 4.10] that, given a ring A and a nonzero A-module E, the trivial extension $A \ltimes E$ is principal if and only if A is principal and E is cyclic with

$$\mathrm{Ann}_A(E) = M_1 \cdots M_n$$

for some idempotent maximal ideals $M_1, \ldots, M_n$ of A. Now, assume $R := A \ltimes A$, where $(A, \mathfrak{m})$ is a chained ring with $\mathfrak{m}^2 = 0$. We proved above that A is necessarily principal. Then, the aforementioned result yields R is principal if and only if $0 = \mathrm{Ann}_A(A) = \mathfrak{m}^2 = \mathfrak{m}$ if and only if A is a field, completing the proof of the theorem. □

For the special case of trivial extensions of local rings by vector spaces over their residue fields, we obtain the following result.

Corollary 2.2. *Let $(A, \mathfrak{m})$ be a local ring, E a nonzero $\frac{A}{\mathfrak{m}}$-vector space, and $R := A \ltimes E$. Then, the following statements are equivalent:*

(1) *R is semi-regular;*
(2) *R is quasi-Frobenius;*
(3) *R is a chained ring;*
(4) *A is a field and $\dim_A E = 1$.*

Proof. Combine Theorem 2.1 with [5, Theorem 3.1(3)] which handles the equivalence (3) ⇔ (4). □

Of relevance to the above corollary is [10, Theorem 10], which established necessary and sufficient conditions for a chained ring to be semi-regular.

A von Neumann regular ring is a reduced semi-regular ring [30, Proposition 2.7]. Matlis noticed that "(von Neumann) regular rings and quasi-Frobenius rings are seen to have a common denominator of definition–they are both extreme examples of semi-regular rings." One may easily appeal to trivial extensions (since these constructions are not reduced) to provide more examples discriminating between von Neumann regularity and semi-regularity, as shown below. Also, recall that the classic examples of quasi-Frobenius rings are semi-simple rings and quotient rings of principal domains modulo nonzero finitely generated ideals. Theorem 2.1 provides, readily, examples of original quasi-Frobenius rings, as shown below.

Example 2.3. $\frac{\mathbb{Z}}{4\mathbb{Z}} \ltimes \frac{\mathbb{Z}}{4\mathbb{Z}}$ is a quasi-Frobenius ring that is neither von Neumann regular nor principal.

Further, one may provide new examples of semi-regular rings. To proceed further, we need to recall the following fact: if S is a multiplicatively closed subset of the trivial extension $R := A \ltimes E$ and $S_o := S \cap A$, then the universal property of localization yields

$$S^{-1}R \cong S_o^{-1}A \ltimes S_o^{-1}E.$$

Example 2.4. Let A be any non-Noetherian von Neumann regular ring (e.g., infinite direct product of fields). Then $R := A \ltimes A$ is a semi-regular ring that is neither von

Neumann regular nor quasi-Frobenius. Indeed, for every $\mathfrak{m} \in \mathrm{Max}(A)$,

$$R_{\mathfrak{m}\ltimes A} = A_{\mathfrak{m}} \ltimes A_{\mathfrak{m}}$$

is semi-regular by Corollary 2.2. Moreover, R is coherent by [16, Remark, p. 55]. By [30, Proposition 2.3], R is semi-regular. However, R is neither von Neumann regular (since not reduced) nor quasi-Frobenius (since not Noetherian).

Recall that semi-regularity is a local property in the class of coherent rings [30, Proposition 2.3]. Outside this class, the question was left open. The next example addresses this question. In this vein, recall that coherence is not a local property. Glaz provided an example of a locally Noetherian ring that is not coherent [16, Example, p. 51]. The next example features also a new locally Noetherian (in fact, locally principal) ring which is not coherent.

Example 2.5. Let k be a field, $A := \prod_{i\in\mathbb{N}} F_i$ and $I := \bigoplus_{i\in\mathbb{N}} F_i$, where $F_i = k\ \forall\ i \in \mathbb{N}$. Then $R := A \ltimes \frac{A}{I}$ is a locally principal quasi-Frobenius ring and, a fortiori, locally semi-regular; which is not coherent and, a fortiori, not semi-regular. Indeed, let P be a prime ideal of R; that is,

$$P := p \ltimes \frac{A}{I}$$

for some prime ideal p of A. Then, we have

$$R_P \cong A_p \ltimes \frac{A_p}{I_p}$$

which is isomorphic to k if $I \nsubseteq p$ or to $k \ltimes k$ if $I \subseteq p$ and, in this case, R_P is a principal quasi-Frobenius ring by Corollary 2.2. Finally, observe that

$$\mathrm{Ann}_R(0, \overline{1}) = I \ltimes \frac{A}{I}$$

is not finitely generated in R since I is not finitely generated in A. So, R is not coherent, as desired.

If A is non-local, then the assumption "$\mathfrak{m}E_t = 0,\ \forall\, \mathfrak{m} \in \mathrm{Max}(A)$" forces $E_t = 0$ and then the hypothesis $E \neq 0$ implies that A is a domain. So, a global version for Theorem 2.1, which also agrees with [3, Theorem 2.10], reads as follows.

Corollary 2.6. *Let A be a domain, E a nonzero torsion-free A-module, and $R := A \ltimes E$. Then, R is semi-regular if and only if A is a field and $E \cong A$.*

3. Transfer of semi-coherence

This section investigates the transfer of semi-coherence in trivial ring extensions issued from rings (i.e., with zero-divisors). The first result establishes the transfer of semi-coherence to trivial extensions over flat modules.

Proposition 3.1. *Let A be a ring, E a nonzero flat A-module, and $R := A \ltimes E$. Then, R is semi-coherent if and only if A is semi-coherent.*

Proof. Notice first that A can be viewed as a subring of $R := A \ltimes E$ and hence R is a flat A-algebra (since E is by hypothesis flat). Assume A is semi-coherent and let M,N be two injective R-modules. Then, by [39, Theorem 3.44], M ($\cong \mathrm{Hom}_R(R,M)$) and N ($\cong \mathrm{Hom}_R(R,N)$) are injective A-modules. Hence, $\mathrm{Hom}_A(M,N)$ is a submodule of a flat A-module F. So, we obtain

$$\mathrm{Hom}_R(M,N) \subseteq \mathrm{Hom}_A(M,N) \subseteq F \subseteq F \otimes_A R$$

where the first containment holds because $A \subseteq R$ and the third containment holds since F is A-flat. Moreover, $F \otimes_A R$ is R-flat. It follows that R is semi-coherent.

Conversely, assume R is semi-coherent and let M,N be two injective A-modules. By the adjoint isomorphism, $\mathrm{Hom}_A(R,M)$ and $\mathrm{Hom}_A(R,N)$ are injective R-modules. Next, consider the following mapping

$$\varphi : \mathrm{Hom}_A(M,N) \longrightarrow \mathrm{Hom}_R\Big(\mathrm{Hom}_A(R,M), \mathrm{Hom}_A(R,N)\Big)$$

defined by $\varphi(u)(f) = u \circ f$, for every $u \in \mathrm{Hom}_A(M,N)$ and $f \in \mathrm{Hom}_A(R,M)$. Clearly, φ is a linear map of A-modules. Moreover, we claim that φ is injective. Indeed, let $u \in \mathrm{Hom}_A(M,N)$ with $\varphi(u) = 0$ and let $x \in M$. Consider the following A-map

$$f : R \longrightarrow M \ ; \ (a,e) \mapsto ax.$$

Then, we have

$$0 = \varphi(u)(f)(1,0) = u(f(1,0)) = u(x)$$

which yields $u = 0$, as desired. By hypothesis, we have

$$\mathrm{Hom}_R\Big(\mathrm{Hom}_A(R,M), \mathrm{Hom}_A(R,N)\Big) \subseteq F$$

where F is a flat R-module, which is also a flat A-module since R is A-flat. Consequently, A is semi-coherent, completing the proof of the result. □

Next, we show how one can use the above results to provide new examples discriminating between the notions of semi-coherence, coherence, and semi-regularity. For this purpose, we first establish a lemma on coherence (which generalizes [25, Theorem 3.1(1)]).

Lemma 3.2. *Let A be a domain, E a torsion free A-module, and $R := A \ltimes E$. Then, R is coherent if and only if A is coherent and E is finitely generated.*

Proof. Assume that R is coherent. Then A, being a retract of R, is coherent by [16, Theorem 4.1.5]. Moreover, let $0 \neq e \in E$. By [16, Theorem 2.3.2(7)], $\mathrm{Ann}_R(0,e)$ is finitely generated. Since E is torsion free, we get

$$\mathrm{Ann}_R(0,e) = 0 \ltimes E.$$

It follows that E is finitely generated. Conversely, assume A is a coherent domain and E is a finitely generated A-module. Then, E is a submodule of a finitely generated free A-module [39, Lemma 4.31], which is then coherent [16, Theorem 2.2.3]. Therefore, E is coherent. It follows that R is coherent by [16, Remark, p. 55]. □

Example 3.3. Let A be a domain which is not a field with quotient field K. Then:

(1) $K \ltimes K^2$ is a coherent ring which is not semi-regular by Lemma 3.2 and Corollary 2.2.
(2) $A \ltimes K$ is a semi-coherent ring which is not coherent by Proposition 3.1 and Lemma 3.2.

Recall that a ring R is mininjective (also called mini-injective) if every R-homomorphism from a simple ideal of R to an R-module extends to R. Harada proved that an Artinian mininjective ring is quasi-Frobenius [21, Theorem 13]. Next, we provide an example which shows that, unlike self semi-injectivity and self fp-injectivity, mininjectivity does not coincide with semi regularity in the class of coherent rings. Moreover, in [30, Proposition 2.2], Matlis proved that, for a ring R, if $Q(R)$ is semi-regular, then R is semi-coherent; and the converse was left open. The example shows that the converse does not hold, in general, even for R coherent.

Example 3.4. Let A be a coherent domain (e.g., Prüfer) and let $R := A \ltimes A^2$. Then:

(1) R is coherent by Lemma 3.2.
(2) R is mininjective by [26, Lemma 3.1 & Theorem 3.3].
(3) $Q(R)$ is not semi-regular. Indeed, one can easily check that $\mathrm{Z}(R) = 0 \ltimes A^2$. Hence, for $S := R \setminus \mathrm{Z}(R)$, we obtain $S_o := S \cap A = A \setminus \{0\}$ and thus
$$Q(R) \cong S_o^{-1}A \ltimes S_o^{-1}A^2 = K \ltimes K^2$$
where $K := Q(A)$. By Example 3.3, $Q(R)$ is not semi-regular, and hence neither is R by [30, Proposition 2.1].

We close this section with observing that the assumption "$\mathfrak{m}E_t = 0$" in Theorem 2.1 is (convenient but) not inevitable in order to construct quasi-Frobenius rings issued from trivial ring extensions, as shown by the next example.

Example 3.5. Let $(A, \mathfrak{m})$ be an Artinian local ring with residue field K and let E denote an injective envelope of K. Then, by Kourki's result [26, Theorem 3.6], $R := A \ltimes E$ is quasi-Frobenius. Indeed, it suffices to verify that the socle of $\mathrm{Ann}_A(E) \times E$ is square free; that is, $\mathrm{Ann}_{\mathrm{Ann}_A(E)\times E}(\mathfrak{m})$ is either null or simple [26, Lemma 3.1]. In fact, we have
$$\mathrm{Ann}_A(E) = 0 \text{ and } \mathrm{Ann}_E(\mathfrak{m}) = K$$

which yield

$$\begin{aligned}\mathrm{Ann}_{\mathrm{Ann}_A(E)\times E}(\mathfrak{m}) &= \mathrm{Ann}_{\mathrm{Ann}_A(E)}(\mathfrak{m})\times \mathrm{Ann}_E(\mathfrak{m})\\ &= \mathrm{Ann}_{(0)}(\mathfrak{m})\times K\\ &= 0\times K, \text{ as desired.}\end{aligned}$$

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