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3D fuzzy data approximation by fuzzy smoothing bicubic splines

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Abstract

Function approximation and interpolation are major and important problems in various scientific fields. In this work we present an approximation method of fuzzy data defined at a 3D fuzzy data set. We define a fuzzy smoothing bicubic spline approximation for a given fuzzy data set and we estimate the approximation error using similarity measures of fuzzy numbers. Examples are given to test the goodness of the method and compare the behavior of the indices proposed for different configurations of the fuzzy smoothing bicubic spline. Finally, some conclusions of the presented method are briefly discussed.

Keywords: Fuzzy data; Approximation theory; Smoothing splines; Variational methods; Bicubic splines

1. Introduction

One of the most interesting and extended problems in applied mathematics is function surface fitting approximation and interpolation. The following interpolation problem of fuzzy data was first proposed by Zadeh, see [23]. Suppose that we have $n+1$ different points in $\mathbb{R}$, and for each of these numbers a fuzzy value in $\mathbb{R}$, u_i, $i = 0, 1, \ldots, n$. Zadeh gave the question whether it is possible to define some kind of smooth function on $\mathbb{R}$ over the $n+1$ points. Lowen investigated a fuzzy Lagrange interpolation in [13]. Later, Kaleva [10] proposed some properties of Lagrange interpolation by using cubic spline approximation. Wang and Li gave the definition of simple fuzzy numbers and the expressions of their membership functions in [22]. There exist different methodologies for approximation to data. For example, the authors in [14,16] gave a method for approximation or fitting surfaces to data, using smoothing splines. Abbasbandy et al. [1] gave a method for finding the best approximation of fuzzy function on a set of points. In [17] a new method for finding best approximation for function by using trapezoidal fuzzy numbers is given.

The authors in [6] proposed a new set of spline functions to interpolate given fuzzy data. Approximation of fuzzy data can be obtained in different research areas. For example, the authors in [5] constructed approximations that contain a sequence of fuzzy numbers by using the F-transform and the max-product Bernstein operators. Later,

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Huang and his co-authors [9] considered how to smooth fuzzy numbers and construct smooth approximations for fuzzy numbers by using the convolution method. In [19] Shu and Wu introduced a new methodology to analyze the quality-based supplier selection and evaluation using fuzzy data from light emitting diodes. In [2], a new set of spline functions denoted as "fuzzy splines" to interpolate fuzzy data is defined.

In the literature, there exist different methodologies for fitness approximation techniques, using multi-objective evolutionary algorithms [15]. Numerical examples are presented to illustrate a new methodology for approximation and interpolation of fuzzy numbers by cubic smoothing spline presented in [20,21].

In this paper a new approximation method of fuzzy data by fuzzy smoothing bicubic splines is presented as an approximation of fuzzy bivariate functions.

The paper is organized as follows: After this introduction, Section 2 presents some notations and preliminaries about C^2-cubic and bicubic B-splines spaces. Section 3 is devoted the definition, computations and convergence results of the smoothing variational bicubic splines. In Section 4 we present the basic definition of the fuzzy numbers and some of the existing similarity measures of fuzzy numbers presented in the bibliography [3,4,7]. In Section 5 the proposed methodology for fuzzy approximation of fuzzy numbers by fuzzy smoothing bicubic spline functions is presented in detail and a convergence result is established. Several experiments are carried out in Section 6 to analyze the behavior of the approximation of fuzzy numbers, performed by fuzzy smoothing bicubic spline functions using different configurations (number of knots, different numbers of approximation points and several values of parameter ε). Then, to analyze individual and global behavior for the different proposed error and similarity indices, statistical analyses are shown. Finally, some conclusions are realized in Section 7.

2. Preliminaries

We denote by $\langle\cdot\rangle_k$ and $\langle\cdot,\cdot\rangle_k$, respectively, the Euclidean norm and inner product in $\mathbb{R}^k$. For any real intervals (a,b) and (c,d), with $a<b$ and $c<d$, we consider the rectangle $\Omega=(a,b)\times(c,d)$ and let $H^3(\Omega;\mathbb{R}^k)$ be the usual Sobolev space of (classes of) functions u belong to $L^2(\Omega;\mathbb{R}^k)$, together with all their partial derivatives $D^\beta(u)$ with $\beta=(\beta_1,\beta_2)$, in the distribution sense, of order $|\beta|=\beta_1+\beta_2\le 3$. For $k=1$ we denote $H^3(\Omega;\mathbb{R}^k)$ by $H^3(\Omega)$. The Sobolev space $H^3(\Omega;\mathbb{R}^k)$ is equipped with the norm

$$\|u\|=\left(\sum_{|\beta|\le 3}\int_\Omega \langle D^\beta u(p)\rangle_k^2 dp\right)^{\frac{1}{2}},$$

the semi-norms

$$|u|_\ell=\left(\sum_{|\beta|=\ell}\int_\Omega \langle D^\beta u(p)\rangle_k^2 dp\right)^{\frac{1}{2}},\quad 0\le\ell\le 3.$$

and the corresponding inner semi-products

$$(u,v)_\ell=\sum_{|\beta|=\ell}\int_\Omega \langle D^\beta u(p),D^\beta v(p)\rangle_k dp,\quad 0\le\ell\le 3.$$

Moreover, for $n,m\in\mathbb{N}^*$, let $\Delta_n=\{x_0,\dots,x_n\}$, $\Delta_m=\{y_0,\dots,y_m\}$ be some subsets of distinct points of $[a,b]$ and $[c,d]$, with $a=x_0<x_1<\cdots<x_n=b$ and $c=y_0<y_1<\cdots<y_m=d$. We denote by $S_3(\Delta_n)$ and $S_3(\Delta_m)$ the spaces of cubic splines of class C^2 given by

$$S_3(\Delta_n)=\{s\in C^2[a,b]:\ s|_{[x_{i-1},x_i]}\in\mathbb{P}_3[x_{i-1},x_i],\ i=1,\dots,n\},$$

$$S_3(\Delta_m)=\{s\in C^2[c,d]:\ s|_{[y_{j-1},y_j]}\in\mathbb{P}_3[y_{j-1},y_j],\ j=1,\dots,m\},$$

where $\mathbb{P}_3[x_{i-1},x_i]$ ($\mathbb{P}_3[y_{j-1},y_j]$) is the restriction on $[x_{i-1},x_i]$ ($[y_{j-1},y_j]$) of the linear space of real polynomials with total degree less than or equal to 3. It is known that dim $S_3(\Delta_n)=n+3$ (dim $S_3(\Delta_m)=m+3$). Let $\{\phi_1,\dots,\phi_{n+3}\}$ and $\{\psi_1,\dots,\psi_{m+3}\}$ be bases of functions with local support of $S_3(\Delta_n)$ and $S_3(\Delta_m)$, respectively and consider the space $S_3(\Delta_n\times\Delta_m)$ of bicubic splines functions of class C^2 given by

$$S_3(\Delta_n\times\Delta_m)=span\ \{\phi_1,\dots,\phi_{n+3}\}\otimes\ span\ \{\psi_1,\dots,\psi_{m+3}\}$$

This space is a Hilbert subspace of $H^3(\Omega)$ equipped with the same norm, semi-norms and inner semi-products of such space and verifies

$$S_3(\Delta_n \times \Delta_m) \subset H^3(\Omega) \cap C^2(\Omega). \tag{1}$$

Particularly, let

$$\{B_0^3(x), \ldots, B_{n+2}^3(x)\} \left(\{B_0^3(y), \ldots, B_{m+2}^3(y)\}\right)$$

be the C^2-cubic B-splines basis of $S_3(\Delta_n)$ $(S_3(\Delta_m))$, then

$$\{B_r^3(x)B_s^3(y),\ r = 0, \ldots, n+2,\ s = 0, \ldots, m+2\}$$

is the C^2-bicubic B-splines basis of $S_3(\Delta_n \times \Delta_m)$, then $dim\ S_3(\Delta_n \times \Delta_m) = (n+3)(m+3)$ and we can define

$$B_k(x, y) = B_r^3(x)B_s^3(y),\ (x, y) \in \Omega,$$

for $r = 0, \ldots, n+2$, $s = 0, \ldots, m+2$, $k = (m+3)r + s + 1$. Then $1 \le k \le (n+3)(m+3)$ and if we denote $M = (n+3)(m+3)$, we have that

$$B_1(x, y), \ldots, B_M(x, y)$$

is a basis of $S_3(\Delta_n \times \Delta_m)$.

3. Smoothing bicubic splines

Let $N \in \mathbb{N}$, $A^N = \{a_1, \ldots, a_N\} \subset \Omega$ and suppose that A^N contains a $\mathbb{P}_2$-unisolvent subset and

$$\sup_{p\in\Omega} \min_{a\in A^N} \langle p - a\rangle_2 = O(\frac{1}{N}),\ N \to +\infty. \tag{2}$$

For $k \in \mathbb{N}^*$ let $B^N = \{b_1, \ldots, b_N\} \subset \mathbb{R}^k$, $\varepsilon > 0$ and let J be the real functional defined on $H^3(\Omega; \mathbb{R}^k)$ by

$$J(v) = \sum_{i=1}^{N} \langle v(a_i) - b_i\rangle_k^2 + \varepsilon |v|_3^2. \tag{3}$$

Then, we consider the following minimization problem: Find $\sigma^N \in (S_3(\Delta_n \times \Delta_m))^k$ such that

$$J(\sigma^N) \le J(v),\ \forall\ v \in (S_3(\Delta_n \times \Delta_m))^k. \tag{4}$$

Theorem 1. *Problem (4) has a unique solution, called the smoothing variational bicubic spline associated with A^N, B^N and ε, which is also the unique solution of the following variational problem: Find $\sigma^N \in (S_3(\Delta_n \times \Delta_m))^k$ such that*

$$\sum_{i=1}^{N} \langle \sigma^N(a_i), v(a_i)\rangle_k + \varepsilon(\sigma^N, v)_3 = \sum_{i=1}^{N} \langle b_i, v(a_i)\rangle_k, \forall\ v \in (S_3(\Delta_n \times \Delta_m))^k. \tag{5}$$

Proof. It is similar to the proof of Theorem 3.1 in [12].

Applying (5) and taking into account that $\sigma^N \in (S_3(\Delta_n \times \Delta_m))^k$ we can write

$$\sigma^N(x, y) = \sum_{i=1}^{M} \alpha_i B_i(x, y),\ \forall\ (x, y) \in \Omega,$$

where $\alpha_i \in \mathbb{R}^k$, $i = 1, \ldots, M$, and $\alpha = (\alpha_1, \ldots, \alpha_M)^t$ is the solution of the linear system

$$(AA^t + \varepsilon R)\alpha = Ab,$$

being $A = \left(B_i(a_j)\right)_{\substack{i=1,\ldots,M,\\ j=1,\ldots,N}}$, $R = \left((B_i, B_j)_3\right)_{i,j=1,\ldots,M}$ and $b = (b_i)_{i=1,\ldots,N}$.

Let $h = \max\{\frac{b-a}{n}, \frac{d-c}{m}\}$.

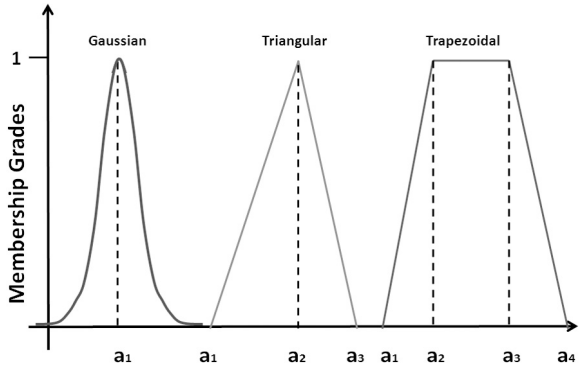

Fig. 1. Examples of Gaussian, triangular and trapezoidal fuzzy numbers.

Theorem 2. *Let $f \in (C^4(\Omega))^k$. Suppose that the hypothesis (2) holds and that*

$$\varepsilon = o(N^2), \ N \to +\infty, \tag{6}$$

$$\frac{Nh^4}{\varepsilon^{\frac{1}{2}}} = o(1), \ N \to +\infty. \tag{7}$$

Then, one has

$$\lim_{N \to +\infty} \|f - \sigma^N\| = 0. \tag{8}$$

Proof. It is analogous to the proof of Theorem 5.3 of [12] taking into account that from (7), $h \to 0$ as $N \to +\infty$.

4. Basic definitions about fuzzy numbers

Definition 3. A fuzzy number is a mapping $u : \mathbb{R} \longrightarrow [0, 1]$ with the following properties (see [11]).

(i) u is an upper semi-continuous function on $\mathbb{R}$.

(ii) $u(x) = 0$ outside some interval $[a_1, a_4] \subset \mathbb{R}$.

(iii) There exist real numbers a_2 and a_3 such that $a_1 \leq a_2 \leq a_3 \leq a_4$ with

- (a) $u(x)$ is a monotonic increasing function on $[a_1, a_2]$,
- (b) $u(x)$ is a monotonic decreasing function on $[a_3, a_4]$,
- (c) $u(x) = 1$, for all $x \in [a_2, a_3]$.

A popular type of fuzzy number is the set of trapezoidal fuzzy numbers, $\mathbb{TFN}$, (see Fig. 1), that can be defined as $A = (a_1, a_2, a_3, a_4)$, and their membership function is defined by

$$\mu_A(x) = \begin{cases} \dfrac{x - a_1}{a_2 - a_1}, & a_1 \leq x \leq a_2, \\ 1, & a_2 \leq x \leq a_3, \\ \dfrac{a_4 - x}{a_4 - a_3}, & a_3 \leq x \leq a_4, \\ 0, & otherwise. \end{cases}$$

If $a_1 = a_2 = a_3 = a_4$, then the real number is represented by a. If $a_1 = a_2$ and $a_3 = a_4$, then A is called a crisp interval. Note that a triangular fuzzy number is obtained when $a_2 = a_3$, (see Fig. 1), in which case triangular fuzzy numbers can be defined by $A = (a_1, a_2, a_3)$.

Definition 4. Let $u = (u_1, u_2, u_3, u_4) \in \mathbb{TFN}$ and $0 < \alpha \leq 1$, then it is called α-cut of u the set

$$[u]^\alpha = \{x \in \mathbb{R} : u(x) \geq \alpha\}.$$

It is defined the 0-cut of u as its support, i.e.,

$$[u]^0 = \bigcup_{0<\alpha\leq 1} [u]^\alpha = [u_1, u_4]$$

Remark 5. An equivalent definition of a trapezoidal fuzzy number $u = (u_1, u_2, u_3, u_4)$ is a function $u : [0, 1] \longrightarrow I$ given by

$$u(\alpha) = [\underline{u}(\alpha), \bar{u}(\alpha)],$$

with

$$\begin{aligned} \underline{u}(\alpha) &= u_1 + (u_2 - u_1)\alpha, \\ \bar{u}(\alpha) &= u_4 + (u_3 - u_4)\alpha, \end{aligned} \tag{9}$$

where I is the set of the all real closed intervals. Obviously we have that $u(\alpha) = [u]^\alpha$, for any $0 \leq \alpha \leq 1$. □

For any $u, v \in \mathbb{TFN}$, $\lambda \in \mathbb{R}$, the sum $u + v$ and the product λu are defined by

$$[u + v]^\alpha = [u]^\alpha + [v]^\alpha, \ [\lambda u]^\alpha = \lambda [u]^\alpha,$$

for all $\alpha \in [0, 1]$, $\lambda > 0$, taking into account that

$$\lambda[\underline{u}(\alpha), \bar{u}(\alpha)] = \begin{cases} [\lambda\underline{u}(\alpha), \lambda\bar{u}(\alpha)], & \lambda \geq 0, \\ [\lambda\bar{u}(\alpha), \lambda\underline{u}(\alpha)], & \lambda < 0. \end{cases}$$

Definition 6. For any $u, v \in \mathbb{TFN}$, it is defined the Hausdorff distance between u and v as the quantity

$$d(u, v) = \sup_{\alpha\in(0,1]} \max\{|\underline{u}(\alpha) - \underline{v}(\alpha)|, |\bar{u}(\alpha) - \bar{v}(\alpha)|\}.$$

Remark 7. From Definition 6, we have

$$d(u, v) = \max_{i=1,2,3,4} |u_i - v_i|, \tag{10}$$

The concept of similarity or dissimilarity between two data sets is fundamental in almost every scientific field. The analysis of similarity or dissimilarity measures between fuzzy sets has gained importance due to the widespread of applications in diverse fields, including fuzzy risk analysis problems [18], decision making [8] and function approximation [4]. In this section, we introduce some existing similarity measures of fuzzy numbers.

Let A and B be two trapezoidal fuzzy numbers, where $A = (a_1, a_2, a_3, a_4)$, $a_1 \leq a_2 \leq a_3 \leq a_4$, and $B = (b_1, b_2, b_3, b_4)$, $b_1 \leq b_2 \leq b_3 \leq b_4$. Then the degree of similarity $S(A, B)$ between the trapezoidal fuzzy numbers A and B is defined in Chen [3] as follows

$$S(A, B) = 1 - \frac{\sum_{i=1}^{4} \mid a_i - b_i \mid}{4},$$

where $\mid a \mid$ is the absolute value of the real number a.

Hsieh and Chen [7] proposed a similarity measure using the *graded mean integration-representation distance* where the degree of similarity $S(A, B)$ between the fuzzy numbers A and B is calculated as follows:

$$S(A, B) = \frac{1}{1 + d(A, B)}$$

where $d(A, B) = \mid P(A) - P(B) \mid$; $P(A)$ and $P(B)$ are the graded mean integration representations of A and B, respectively. If A and B are trapezoidal fuzzy numbers, then the graded mean integration of these fuzzy numbers is defined as:

$$P(A) = \frac{a_1 + 2a_2 + 2a_3 + a_4}{6},$$

$$P(B) = \frac{b_1 + 2b_2 + 2b_3 + b_4}{6}.$$

In [4], Chen et al. introduced a new method called the simple center of gravity method (denoted as SCGM) to calculate the center of gravity points (x_A^*, y_A^*) and (x_B^*, y_B^*) of generalized fuzzy numbers A and B respectively. If A and B are two trapezoidal fuzzy numbers, the degree of similarity $S(A, B)$ between these numbers can be calculated as follows:

$$S(A, B) = 1 - \frac{\sum_{i=1}^{4} \mid a_i - b_i \mid}{4} \times \left(1 - \mid x_A^* - x_B^* \mid\right)^{B(S_A, S_B)} \times \frac{min(y_A^*, y_B^*)}{max(y_A^*, y_B^*)},$$

where $S(A, B) \in [0, 1]$, and

$$x_A^* = \frac{y_A^*(a_3 + a_2) + (a_4 + a_1)(1 - y_A^*)}{2},$$

$$y_A^* = \begin{cases} \frac{1}{2}, & if \quad a_1 = a_4, \\ \frac{1}{6}(\frac{a_3 - a_2}{a_4 - a_1} + 2), & if \quad a_1 \neq a_4, \end{cases}$$

and

$$B(S_A, S_B) = \begin{cases} 1, & if \quad S_A + S_B > 0, \\ 0, & if \quad S_A + S_B = 0, \end{cases}$$

where S_A and S_B are the lengths of the bases of trapezoidal fuzzy numbers A and B, respectively, and defined by:

$$S_A = a_4 - a_1,$$
$$S_B = b_4 - b_1.$$

In [24], to make the similarity well distributed, a new method SIAM (Shape's Indifferent Area and Midpoint) to measure triangular fuzzy number is put forward, which takes the shape's indifferent area and midpoint of two triangular fuzzy numbers into consideration.

The goal of the present paper is to show the effectiveness of a new fuzzy numbers approximation method and not the similarity indexes so the choice of these is not essential in our work.

Definition 8. A fuzzy function defined on $\Omega \subset \mathbb{R}^2$ into the trapezoidal fuzzy number set $\mathbb{TFN}$ is an application $f : \Omega \to \mathbb{TFN}$ such that $f = (f_1, f_2, f_3, f_4)$, where f_i is a real function defined on Ω, for $i = 1, 2, 3, 4$, and $f(x, y) \in \mathbb{TFN}$, for any $(x, y) \in \Omega$.

Definition 9. The fuzzy bicubic spline space constructed on the partition $\Delta_n \times \Delta_m$ of Ω is the set of the fuzzy functions

$$S_3(\Delta_n \times \Delta_m; \mathbb{TFN}) = \{s : \Omega \to \mathbb{TFN} : s = \sum_{i=1}^{M} \alpha_i B_i, \ \alpha_i \in \mathbb{TFN}, \ i = 1, \ldots, M\}.$$

5. Fuzzy smoothing bicubic splines

Now, we consider the following approximation problem: Given $U^N = \{u_1, \ldots, u_N\} \subset \mathbb{TFN}$ find a fuzzy function $s \in S_3(\Delta_n \times \Delta_m; \mathbb{TFN})$ such that $s(a_\ell) \approx u_\ell$, for any $\ell = 1, \ldots, N$.

We can consider that U^N is a subset of $\mathbb{R}^4$ since $u_\ell = (u_{\ell 1}, u_{\ell 2}, u_{\ell 3}, u_{\ell 4}) \in \mathbb{R}^4$, for any $\ell = 1, \ldots, N$.

Let σ^N be the smoothing bicubic variational spline associated with A^N, U^N and $\varepsilon > 0$ defined in Theorem 1. Then $\sigma^N \in (S_3(\Delta_n \times \Delta_m))^4$ and thus there exist $\alpha_1, \ldots, \alpha_M \in \mathbb{R}^4$ such that

$$\sigma^N(x, y) = \sum_{i=1}^{M} \alpha_i B_i(x, y), \ \forall (x, y) \in \Omega.$$

Consider a fuzzy function $f : \Omega \to \mathbb{TFN}$ and $U^N = f(A^N) = \{f(a_\ell) : \ell = 1, \dots, N\}$. For all $i = 1, \dots, M$, let $\overline{\alpha}_i \in \mathbb{R}^4$ such that its components are the same as those of α_i ordered from lowest to highest, i.e., if $\alpha_i = (\alpha_{i1}, \alpha_{i2}, \alpha_{i3}, \alpha_{i4})$ and $\overline{\alpha}_i = (\overline{\alpha}_{i1}, \overline{\alpha}_{i2}, \overline{\alpha}_{i3}, \overline{\alpha}_{i4})$ then

$$\overline{\alpha}_{ij} = \alpha_{i\gamma(j)}, \ \ j = 1, 2, 3, 4,$$

being $\gamma : \{1, 2, 3, 4\} \to \{1, 2, 3, 4\}$ the permutation such that

$$\overline{\alpha}_{i1} \le \overline{\alpha}_{i2} \le \overline{\alpha}_{i3} \le \overline{\alpha}_{i4}.$$

Then $\overline{\alpha}_i \in \mathbb{TFN}$, $i = 1, \dots, M$ and thus the function s^N given by

$$s^N(x, y) = \sum_{i=1}^{M} \overline{\alpha}_i B_i(x, y), \ \forall (x, y) \in \Omega,$$

verifies that $s^N \in S_3(\Delta_n \times \Delta_m; \mathbb{TFN})$.

This fuzzy function is called the fuzzy smoothing bicubic spline associated with A^N, $f(A^N)$ and ε.

Theorem 10. *Consider a fuzzy function $f : \Omega \to \mathbb{TFN}$ such that $f = (f_1, f_2, f_3, f_4)$ and $f_i \in C^4(\Omega)$ and suppose the hypotheses (2), (6) and (7) hold. Then, one has*

$$\lim_{N \to +\infty} \overline{S}(f, s^N) = 1, \tag{11}$$

being $\overline{S}(f, s^N) = \frac{1}{Z}\sum_{i=1}^{Z} S(f(\xi_i), s^N(\xi_i))$, where $\{\xi_1, \dots, \xi_Z\} \subset \Omega$ is a set of Z random points of the domain Ω and S is the Chen (S_{CHEN}) index, the Hsieh index (S_{HSIEH}) or the Chen & Chen index (S_{SCGM}), defined in Section 4. *Moreover*

$$\lim_{N \to +\infty} d(f(p), s^N(p)) = 0, \ \forall\, p \in \Omega. \tag{12}$$

Proof. For any $N \in \mathbb{N}$ let s^N be the fuzzy smoothing bicubic spline associated with A^N, $f(A^N)$ and ε. Let d_N the real number given by

$$d_N = \max_{p \in \Omega} \max_{i=1,2,3,4} f_{i+1}(p) - f_i(p).$$

Let σ^N be the smoothing bicubic variational spline associated with A^N, $f(A^N) \subset \mathbb{R}^4$ and ε, considering $f(a_\ell)$ as an element of $\mathbb{R}^4$, for any $\ell = 1, \dots, N$.

From Theorem 2 we can deduce that there exists $N_0 \in \mathbb{N}$ such that for $N > N_0$, $i = 1, 2, 3, 4$ and $p \in \Omega$ we have

$$|f_i(p) - \sigma_i^N(p)| < \frac{d_N}{2}.$$

Thus, for $i = 1, 2, 3$, we obtain that

$$\sigma_i^N(p) - \sigma_{i+1}^N(p) = \sigma_i^N(p) - f_i(p) + f_i(p) - f_{i+1}(p) + f_{i+1}(p) - \sigma_{i+1}^N(p) \le 0.$$

Then, for $N > N_0$, $\sigma_1^N(p) \le \sigma_2^N(p) \le \sigma_3^N(p) \le \sigma_4^N(p)$ and thus $\sigma^N(p) \in \mathbb{TFN}$, for any $p \in \Omega$. Hence $s^N = \sigma^N$ and from Theorem 2

$$\lim_{N \to +\infty} |f_i(p) - s_i^N(p)| = 0, \ \forall\, p \in \Omega, \ i = 1, 2, 3, 4,$$

and we can confirm that (11) and (12) hold.

6. Numerical examples

In this section, different interpolation error and similarity estimations are proposed in order to analyze the presented fuzzy interpolation method. The definition of these estimations is

$$\overline{S} = \frac{1}{Z} \sum_{i=1}^{Z} S(f(\xi_i), \sigma(\xi_i)), \tag{13}$$

Table 1
Function $f(x, y)$. Error and similarity indices estimates for different values of the approximation method parameters.

$n = m$	N	ε	$\overline{S}_d$	$\overline{S}_{CHEN}$	$\overline{S}_{HSIEH}$	$\overline{S}_{SCGM}$
5	50	10^{-5}	7.4742×10^{-2}	0.9626	0.9845	0.9579
5	50	10^{-7}	2.4238×10^{-2}	0.9846	0.9866	0.9598
5	50	10^{-9}	3.7776×10^{-2}	0.9883	0.9568	0.9300
5	100	10^{-5}	9.5961×10^{-2}	0.9946	0.9919	0.9764
5	100	10^{-7}	4.7167×10^{-2}	0.9956	0.9921	0.9783
5	100	10^{-9}	4.8663×10^{-2}	0.9839	0.9850	0.9710
5	500	10^{-5}	6.7796×10^{-2}	0.9682	0.9940	0.9845
5	500	10^{-7}	9.6577×10^{-3}	0.9939	0.9943	0.9838
5	500	10^{-9}	2.9117×10^{-2}	0.9894	0.9941	0.9821
9	50	10^{-5}	2.7781×10^{-2}	0.9883	0.9910	0.9753
9	50	10^{-7}	2.0309×10^{-2}	0.9907	0.9921	0.9785
9	50	10^{-9}	3.7296×10^{-2}	0.9898	0.9917	0.9818
9	100	10^{-5}	3.2391×10^{-2}	0.9826	0.9860	0.9900
9	100	10^{-7}	1.3013×10^{-2}	0.9955	0.9918	0.9925
9	100	10^{-9}	6.0140×10^{-2}	0.9818	0.9841	0.9890
9	500	10^{-5}	4.1107×10^{-3}	0.9985	0.9994	0.9984
9	500	10^{-7}	2.1545×10^{-3}	0.9986	0.9996	0.9990
9	500	10^{-9}	2.8174×10^{-3}	0.9985	0.9995	0.9898
17	50	10^{-5}	7.4268×10^{-2}	0.9679	0.9898	0.9720
17	50	10^{-7}	1.5358×10^{-2}	0.9920	0.9913	0.9743
17	50	10^{-9}	2.2139×10^{-2}	0.9894	0.9912	0.9713
17	100	10^{-5}	6.7632×10^{-3}	0.9962	0.9976	0.9931
17	100	10^{-7}	1.8586×10^{-3}	0.9988	0.9983	0.9957
17	100	10^{-9}	8.0263×10^{-3}	0.9977	0.9982	0.9956
17	500	10^{-5}	3.7018×10^{-4}	0.9998	0.9998	0.9960
17	500	10^{-7}	8.2134×10^{-5}	0.99997	0.99991	0.99983
17	500	10^{-9}	7.1242×10^{-5}	0.99998	0.99992	0.99985

where $\{\xi_1, \ldots, \xi_Z\} \subset R$ is a set of Z random points in the domain R and S is the Chen (S_{CHEN}) index, the Hsieh index (S_{HSIEH}), the Chen & Chen index (S_{SCGM}), defined in Section 4, or the Hausdorff distance (d) given in Definition 6. From Theorem 10, it should be verified that $\overline{S}$ tends to 1 as $N \to +\infty$, for $S = S_{CHEN}, S_{HSIEH}, S_{SCGM}$, and $\overline{S}$ tends to 0 as $N \to +\infty$, for $S = d$.

To test our method we consider an example for different partitions of the domain, different numbers of knots and different values of smoothing parameter ε for the function $f : [0, \pi] \times [0, \pi] \longrightarrow \mathbb{TFN}$ given by

$$f(x, y) = (0.2\exp(-0.5y)\sin(5x) + 0.5, 0.1\exp(-0.5x)\sin(5y) + 0.3,$$
$$0.2\exp(-0.5y)\sin(3x), 0.1\exp(-0.5y)\sin(x) + 0.3\exp(-0.6y)\cos(5y) - 0.4).$$

For the simulations presented in this section, the number of points to compute the estimation $\overline{S}$ given by (13) is $Z = 500$ in all cases.

Table 1 illustrates the performance of the fuzzy smoothing bicubic spline for different values of the knot numbers n and m, taking $n = m$ in all cases, the approximation points number and the smoothing parameter ε. The proposed simulations show the influence and relative importance of these parameter values in the effectiveness of the approximation. Specifically, under the hypotheses of Theorem 10, the error estimation $\overline{S}_d$ decreases to 0 and the similarity index estimations increase to 1 as $N \to +\infty$. Finally, we observe that if M and N are fixed then the error and similarity estimations are not monotones respect to ε; then we can surmise the existence of an optimal value of ε.

7. Conclusion

In this paper, we have extended the methodology in the case of univariate cubic spline presented in [20] for approximation error of 3D fuzzy data by using fuzzy smoothing bicubic splines.

Through various experiments with a two-dimensional fuzzy function, we have modified the number of knots ($(n + 1) \times (m + 1)$), the number of points used for the smoothing spline system (M) and the parameter ε used for the tradeoff between precision and smoothness. The developed numerical examples illustrate the relevance of the approximation method presented in this work and it confirms the established convergence result.

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