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ORIGINAL PAPER

Approximation of fuzzy functions by fuzzy interpolating bicubic splines: 2018 CMMSE conference

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Abstract

One of the most interesting and important techniques in applied mathematics is interpolation and approximation. It can be also shown that basic theory of fuzzy sets and operations with fuzzy numbers can be used, and result very convenient, for solving many analytical chemistry problems: as depth profile comparison or calibration with errors in signals and/or concentrations, for spectra interpretation, or even for automatic qualitative analysis or expert systems with X-ray spectroscopy. In this special context, we introduce in this paper a new fuzzy interpolation method of fuzzy data or functions, specially indicated in all these chemical applications. We will use bicubic splines (cubic splines of two variables) of class $\mathcal{C}^2$ as linear combinations of a basis constructed by a tensor product of univariate B-splines in each of these variables. First we establish a bicubic interpolation spline method of a given fuzzy data set or a given fuzzy bivariate function. We proof the existence of a unique solution of this problem and we show a convergence result. Finally, we test the effectiveness of this method by some numerical and graphical examples.

Keywords Interpolation and approximation · Fuzzy data · Bicubic splines · Analytical chemistry · Calibration · Spectroscopy

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1 Introduction

Function approximation and interpolation are essential problems in almost every scientific field. Given a set of multiple input single output data, with input data $X = \{x_0, \ldots, x_n\}$, and output data Y, the main goal of function approximation is obtaining a model to approximate the dependent variable Y, given the input variable X, being X and Y real number sets. The interpolation problem of fuzzy data was first introduced by Zadeh [20] and can be formulated as "suppose we are given $n+1$ points $x_0, \ldots, x_n \in \mathbb{R}$ and for each of these points a Fuzzy Value in $\mathbb{R}$; then, it is possible to construct some function on $\mathbb{R}$, rather than a crisp one to define some kind of smooth function in $\mathbb{R}$ with the given $n + 1$ points?". Lowen in [15] gave a fuzzy Lagrange interpolation theorem, then Kaleva presented some properties of Lagrange and cubic spline interpolation in [10]. Abbasbandy et al. presented in [1] a numerical approximation of fuzzy functions by fuzzy polynomials and found the best approximation for fuzzy functions by optimization to obtain a fuzzy polynomial. A novel methodology for modeling uncertain data with fuzzy B-splines is presented in Anile et al. [3]. In [2] interpolation of fuzzy data by using fuzzy splines is proposed in order to find new set of spline functions called Fuzzy Splines to interpolate fuzzy data. Valenzuela and Pasadas in [19] define new error and similarity indices to determine the accuracy of interpolation of fuzzy data by cubic spline functions.

In this paper a new interpolation method of fuzzy data by fuzzy bicubic splines is presented as an approximation method of bivariate fuzzy functions. As already mentioned, this fuzzy interpolation procedure could be useful and worhtwile for posing and solving different practical problems in different fields of analytical chemistry, as for example: library searching in the infrared and ultraviolet spectral range, chromatographic analysis of urine samples to nephritis classification, gasolines classification based on capillary gas chromatography, for calibration of linear and non-linear signal concentration dependencies, for spectrophotometric multicomponent analysis, and many others.

In fact, many times the chemist becomes aware of the fact that he has to deal with many types of vague, incomplete or inexact data or information, and that the uncertainty of those cannot always be described by means of statistical terms, but can be taken into account by means of the fuzzy theory introduced by Zadeh [20] in 1965. Nowadays, this mathematical theory is a very mature and important branch of Mathematics and Computer Science, with a wide ranging collection of concepts, techniques and applications in almost all branches of general Science and Engineering, and in particular in Analytical and Formal Chemistry. Sometimes a molecule may be regarded as a graph, and all the intrinsecal uncertainty of its formal description can be included in a rigorous framework, via the fuzzy numbers and their corresponding arithmetics and logic. In this new context, fuzzy radius and bond length for atoms become familiar and completely feasible within this theory. In this way, the analytical chemist is able to consider and solve more and more complex problems and answer questions about his own research studies and those raised for the industry and/or society interests.

In principle, some of these procedures are not always limited to the consideration of a single variable but can be also used in the several variables framework. For instance, in chromatography, the retention position and the signal of a peak could be used for classifying unknown samples with a fuzzy method. In such cases, a two-dimensional membership function would be needed (based on a circle, elipse or some trapezoidal piramide). Over these domains of influence (or fuzzy supports) the membership functions are specified as surfaces of suitable structure. Further applications of comparing fuzzy functions are known for peak tracking in high-performance liquid chromatographic separation and for depth-profiling in secondary ion mass spetrometry, see [16] and the references therein.

The paper is organized as follows: after this introduction, in Sect. 2 we briefly recall some preliminary notation and results. In Sect. 3, we explain the bicubic spline interpolation methods. Section 4 briefly presents some basic fundamentals and definitions of fuzzy numbers; in Sect. 5, we explain the proposed methodology of fuzzy interpolating bicubic splines. The convergence of the method is established in Sects. 6 and 7 introduces some of the similarity measures of fuzzy numbers frequently used in the field of fuzzy data. In Sect. 8, different simulation results are carried out showing the good performance of the proposed error and similarity indices. Finally, the conclusions are discussed in Sect. 9.

2 Preliminaries

We denote by $< \cdot >_n$ and $< \cdot, \cdot >_n$, respectively, the Euclidean norm and inner product in $\mathbb{R}^n$. For any real intervals (a,b) and (c,d), with $a < b$ and $c < d$, we consider the rectangle $R = (a, b) \times (c, d)$ and let $H^3(R)$ be the usual Sobolev space of (classes of) functions u belonging to $L^2(R)$, together with all their partial derivatives $D^{\boldsymbol{\beta}}(u)$ with $\boldsymbol{\beta} = (\beta_1, \beta_2)$, in the distribution sense, of order $|\boldsymbol{\beta}| = \beta_1 + \beta_2 \leq 3$. this space is equipped with the norm

$$\|u\| = \left(\sum_{|\boldsymbol{\beta}| \leq 3} \int_R (D^{\boldsymbol{\beta}} u(p))^2 dp \right)^{\frac{1}{2}},$$

the seminorms

$$|u|_\ell = \left(\sum_{|\boldsymbol{\beta}| = \ell} \int_R (D^{\boldsymbol{\beta}} u(p))^2 dp \right)^{\frac{1}{2}}, \quad 0 \leq \ell \leq 3.$$

and the corresponding inner semiproducts

$$(u, v)_\ell = \sum_{|\boldsymbol{\beta}| = \ell} \int_R D^{\boldsymbol{\beta}} u(p) D^{\boldsymbol{\beta}} v(p) dp, \quad 0 \leq \ell \leq 3.$$

Moreover, for $n, m \in \mathbb{N}^*$, let $T_n = \{x_0, \ldots, x_n\}$, $T_m = \{y_0, \ldots, y_m\}$ be some subsets of distinct points of $[a, b]$ and $[c, d]$, with $a = x_0 < x_1 < \ldots < x_n = b$ and

$c = y_0 < y_1 < \ldots < y_m = d$. We denote by $S_3(T_n)$ and $S_3(T_m)$ the spaces of cubic splines of class $\mathcal{C}^2$ given by

$$S_3(T_n) = \{s \in \mathcal{C}^2[a,b] : \; s|_{[x_{i-1},x_i]} \in \mathbb{P}_3[x_{i-1},x_i], \; i = 1,\ldots,n\},$$
$$S_3(T_m) = \{s \in \mathcal{C}^2[c,d] : \; s|_{[y_{j-1},y_j]} \in \mathbb{P}_3[y_{j-1},y_j], \; j = 1,\ldots,m\},$$

where $\mathbb{P}_3[x_{i-1},x_i]$ ($\mathbb{P}_3[y_{j-1},y_j]$) is the restriction on $[x_{i-1},x_i]$ ($[y_{j-1},y_j]$) of the linear space of real polynomials with total degree less than or equal to 3. It is known that $\dim S_3(T_n) = n+3$ ($\dim S_3(T_m) = m+3$). Let $\{\phi_1,\ldots,\phi_{n+3}\}$ and $\{\psi_1,\ldots,\psi_{m+3}\}$ be bases of functions with local support of $S_3(T_n)$ and $S_3(T_m)$ respectively, and consider the space $S_3(T_n \times T_m)$ of bicubic spline functions of class $\mathcal{C}^2$ given by

$$S_3(T_n \times T_m) = span\ \{\phi_1,\ldots,\phi_{n+3}\} \otimes \ span\ \{\psi_1,\ldots,\psi_{m+3}\}$$

This space is a Hilbert subspace of $H^3(R)$ equipped with the same norm, semi-norms and inner semi-products of such space, and verifies

$$S_3(T_n \times T_m) \subset H^3(R) \cap \mathcal{C}^2(R). \tag{1}$$

Particulary, let

$$\{B_0^3(x),\ldots,B_{n+2}^3(x)\} \; \left(\left\{B_0^3(y),\ldots,B_{m+2}^3(y)\right\}\right)$$

be the $\mathcal{C}^2-$cubic B-splines basis of $S_3(T_n)$ ($S_3(T_m)$), then

$$\{B_r^3(x)B_s^3(y), \; r = 0,\ldots,n+2, \; s = 0,\ldots,m+2\}$$

is the $\mathcal{C}^2-$bicubic B-splines basis of $S_3(T_n \times T_m)$, then $dim\ \ S_3(T_n \times T_m) = (n+3)(m+3)$ and we can define

$$B_k(x,y) = B_r^3(x)B_s^3(y), \; (x,y) \in R,$$

for $r = 0,\ldots,n+2$, $s = 0,\ldots,m+2$, $k = (m+3)r+s+1$. Then $1 \leq k \leq (n+3)(m+3)$ and if we denote $M = (n+3)(m+3)$, we have that

$$B_1(x,y),\ldots,B_M(x,y)$$

is the $\mathcal{C}^2-$bicubic B-splines basis of $S_3(T_n \times T_m)$.

3 Interpolating bicubic splines

Let $A^N = \{(x_i,y_j) \in T_n \times T_m, \; i = 0,\ldots,n, \; j = 0,\ldots,m\}$, with $N = (n+1)(m+1)$ and suppose that

$$\sup_{p \in R} \min_{\boldsymbol{a} \in A^N} < \boldsymbol{p} - \boldsymbol{a} >_2 = O\left(\frac{1}{N}\right), \ N \longrightarrow +\infty. \tag{2}$$

From (2) we deduce that $n \longrightarrow +\infty$ and $m \longrightarrow +\infty$ when $N \longrightarrow +\infty$.
Let L_1^N be a Lagrangian operator defined from $H^3(R)$ into $\mathbb{R}^N$ given by

$$L_1^N v = (v(\boldsymbol{a}))_{\boldsymbol{a} \in A^N} \tag{3}$$

and $L_2^N : H^3(R) \longrightarrow \mathbb{R}^{2n+2m+8}$ given by.

$$L_2^N v = (\mathcal{L}_\ell v)_{\ell=1,\ldots,2n+2m+8}, \tag{4}$$

where

$$\mathcal{L}_\ell v = \begin{cases} \dfrac{\partial^2 v}{\partial y^2}(x_{\ell-1}, c), & \ell = 1, \ldots, n+1, \\ \dfrac{\partial^2 v}{\partial y^2}(x_{\ell-n-2}, d), & \ell = n+2, \ldots, 2n+2, \\ \dfrac{\partial^2 v}{\partial x^2}(a, y_{\ell-2n-3}), & \ell = 2n+3, \ldots, 2n+m+3, \\ \dfrac{\partial^2 v}{\partial x^2}(b, y_{\ell-2n-m-4}), & \ell = 2n+m+4, \ldots, 2n+2m+4, \\ \dfrac{\partial^4 v}{\partial x^2 \partial y^2}(x_{in}, y_{jm}), & i = 0, 1, \ j = 0, 1, \ \ell = 2n+2m+4+2i+j+1, \end{cases}$$

Let $B^N = \{u_\ell, \ \ell = 1, \ldots, N\} \subset \mathbb{R}$. It is easy to prove the following result.

Theorem 1 *There exists a unique $S_N \in S_3(T_n \times T_m)$ such that*

$$L_1^N S_N = (u_\ell)_{\ell=1,\ldots,N},$$
$$L_2^N S_N = \mathbf{0} \in \mathbb{R}^{2n+2m+8}$$

called the interpolating natural $\mathcal{C}^2$−bicubic spline associated with A^N and B^N.

Thus $\mathcal{C}^2$-bicubic spline verifies that

$$S_N(x, y) = \sum_{k=1}^{M} \alpha_k B_k(x, y), \ (x, y) \in R,$$

where $\boldsymbol{\alpha} \equiv (\alpha_1, \ldots, \alpha_M)^\top \in \mathbb{R}^M$ is the solution of the linear system

$$\boldsymbol{A}\boldsymbol{\alpha} = \boldsymbol{b}, \tag{5}$$

with $\boldsymbol{A} = \begin{pmatrix} \boldsymbol{A_1} \\ \boldsymbol{A_2} \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} \boldsymbol{b_1} \\ \boldsymbol{b_2} \end{pmatrix}$, where

$$\boldsymbol{A_i} = \left(L_i^N B_k\right)_{k=1,\ldots,M}, \quad (i = 1, 2) \tag{6}$$

$$\boldsymbol{b_1} = (u_\ell)_{\ell=1,\ldots,N}, \tag{7}$$

$$\boldsymbol{b_2} = (0)_{\ell=1,\ldots,M-N}. \tag{8}$$

Following the techniques used in [17] it follows the following result.

Theorem 2 *Let $f \in \mathcal{C}^4(R)$ and let S_N be the interpolating natural $\mathcal{C}^2$-bicubic spline associated with A^N and $L_1^N f$, then there exists a constant $C > 0$ such that*

$$| f - S_N |_\ell \leq C h^{4-\ell}, \quad \ell = 0, 1, 2, 3, \ N \to +\infty, \tag{9}$$

where $h = max \left\{ \frac{b-a}{n}, \frac{d-c}{m} \right\}$. Hence

$$\lim_{N \to +\infty} \| f - S_N \| = 0. \tag{10}$$

4 Fuzzy numbers

When a spectroscopic line has to be identified in order to specify a functional group in infrared spectroscopy or to decide on the presence of an element in atomic spectroscopy, it is necessary to compare the line position with other lines appearing in certain library of reference lines (see [5,16] and references therein). As the experimentally obtained line will surely not match exactly any of the tabulated ones, usually an interval around the reference lines has to be considered in order to decide whether the experimental line coincides with the reference candidate or not. In such a way that a value of 1 is assigned to a line that matches the interval around this reference line and a value of 0 is assigned to lines outside this reference interval. So, only a yes/no answer would be obtained with this crisp (1/0) procedure, and no difference could be made with respect to a line that comes close to the borders of the interval compared with a line that matches exactly the reference line!

With the concept of fuzzy numbers, as we will see in the definitions below, the coincidence or not of these experimental and reference lines can be described in a much more convenient way, and the advantages of fuzzy modelling and fuzzy pattern recognition have demostrated many worthwhile theoretical and practical applications in Formal and Analytical Chemistry (Fig. 1).

Definition 1 A fuzzy number is a mapping $u : \mathbb{R} \longrightarrow [0, 1]$ with the following properties (see [12]).

i) u is an upper semi-continuous function on $\mathbb{R}$.
ii) $u(x) = 0$ outside some interval $[a_1, a_4] \subset \mathbb{R}$.
iii) There exist real numbers a_2 and a_3 such that $a_1 \leq a_2 \leq a_3 \leq a_4$ with

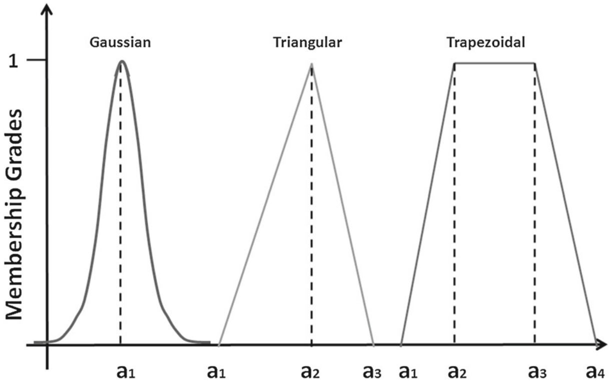

Fig. 1 Examples of gaussian, triangular and trapezoidal fuzzy numbers

a) $u(x)$ is a monotonic increasing function on $[a_1, a_2]$,
b) $u(x)$ is a monotonic decreasing function on $[a_3, a_4]$,
c) $u(x) = 1$, for all $x \in [a_2, a_3]$.

A popular type of fuzzy number is the set of trapezoidal fuzzy numbers, $\mathbb{TFN}$, (see Fig. 3), that can be defined as $\boldsymbol{a} = (a_1, a_2, a_3, a_4)$, and their membership function is defined by

$$\mu(\boldsymbol{a}) = \begin{cases} \dfrac{x - a_1}{a_2 - a_1}, & a_1 \leq x \leq a_2, \\ 1, & a_2 \leq x \leq a_3, \\ \dfrac{a_4 - x}{a_4 - a_3}, & a_3 \leq x \leq a_4, \\ 0, & otherwise. \end{cases}$$

If $a_1 = a_2 = a_3 = a_4 = a$ then the real number is presented by a. If $a_1 = a_2$ and $a_3 = a_4$, then $\boldsymbol{a}$ is called a crisp interval. Note that a triangular fuzzy number is obtained when $a_2 = a_3$, (see Fig. 3), in which case triangular fuzzy numbers can be defined by $\boldsymbol{a} = (a_1, a_2, a_3)$ and their membership function is defined by

$$\mu(\boldsymbol{a}) = \begin{cases} \dfrac{x - a_1}{a_2 - a_1}, & a_1 \leq x \leq a_2, \\ \dfrac{a_3 - x}{a_3 - a_2}, & a_2 \leq x \leq a_3, \\ 0, & otherwise. \end{cases}$$

Definition 2 Let $\boldsymbol{u} = (u_1, u_2, u_3, u_4) \in \mathbb{TFN}$ and $0 < \alpha \leq 1$, then it is called $\alpha-$cut of $\boldsymbol{u}$ the set

$$[\boldsymbol{u}]^\alpha = \{x \in \mathbb{R} : u(x) \geq \alpha\}.$$

It is defined the $0-$cut of u as its support, i.e.,

$$[\boldsymbol{u}]^0 = \underset{0<\alpha\leq 1}{\cup} [\boldsymbol{u}]^\alpha = [u_1, u_4]$$

Remark 1 An equivalent definition of a trapezoidal fuzzy number $\boldsymbol{u} = (u_1, u_2, u_3, u_4)$ is a function $u : [0, 1] \longrightarrow I$ given by

$$u(\alpha) = [\underline{u}(\alpha), \bar{u}(\alpha)],$$

with

$$\begin{aligned} \underline{u}(\alpha) &= u_1 + (u_2 - u_1)\alpha, \\ \bar{u}(\alpha) &= u_4 + (u_3 - u_4)\alpha, \end{aligned} \tag{11}$$

where I is the set of the all real closed intervals. Obviously we have that $u(\alpha) = [\boldsymbol{u}]^\alpha$, for any $0 \leq \alpha \leq 1$. □

For any $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{TFN},\ \lambda \in \mathbb{R}$, the sum $\boldsymbol{u} + \boldsymbol{v}$ and the product $\lambda\boldsymbol{u}$ are defined by

$$[\boldsymbol{u} + \boldsymbol{v}]^\alpha = [\boldsymbol{u}]^\alpha + [\boldsymbol{v}]^\alpha,\ [\lambda\boldsymbol{u}]^\alpha = \lambda[\boldsymbol{u}]^\alpha,$$

for all $\alpha \in [0, 1],\ \lambda > 0$, taking into account that

$$\lambda[\underline{u}(\alpha), \bar{u}(\alpha)] = \begin{cases} [\lambda\underline{u}(\alpha), \lambda\bar{u}(\alpha)], & \lambda \geq 0, \\ [\lambda\bar{u}(\alpha), \lambda\underline{u}(\alpha)], & \lambda < 0. \end{cases}$$

Definition 3 For any $u, v \in \mathbb{TFN}$, it is defined the Haussdorf distance between $\boldsymbol{u}$ and $\boldsymbol{u}$ as the quantity

$$d(\boldsymbol{u}, \boldsymbol{u}) \equiv S_d(\boldsymbol{u}, \boldsymbol{u}) = \underset{\alpha\in(0,1]}{sup}\ max\{|\underline{u}(\alpha) - \underline{v}(\alpha)|, |\bar{u}(\alpha) - \bar{v}(\alpha)|\}. \tag{12}$$

Definition 4 A fuzzy bivariate function defined on the set $R \subset \mathbb{R}^2$ is an application $\boldsymbol{f} : R \longrightarrow \mathbb{TFN}$ such that $\boldsymbol{f} = (f_1, f_2, f_3, f_4)$, where f_i is a real function defined on R and $\boldsymbol{f}(x, y) \in \mathbb{TFN}$, for any $(x, y) \in R$.

5 Fuzzy interpolating bicubic splines

Suppose given two partitions $T_n = \{a = x_0 < x_1, \ldots, < x_n = b\}$ and $T_m = \{c = y_0 < y_1, \ldots, < y_m = d\}$ of $[a, b], [c, d] \subset \mathbb{R}$, respectively, and let $S_3(T_n \times T_m)$ be the corresponding $\mathcal{C}^2-$bicubic spline space and $R = [a, b] \times [c, d]$. Let $N = (n + 1)(m + 1)$ and $M = (n + 3)(m + 3)$ and let $\{B_1(x, y), \ldots, B_M(x, y)\}$ be the $\mathcal{C}^2-$bicubic B-spline basis of $S_3(T_n \times T_m)$.

Definition 5 The fuzzy bicubic splines space constructed on the partition $T_n \times T_m$ is the set of fuzzy functions

$$S_3(T_n \times T_m; \mathbb{TFN}) \equiv$$
$$\{\boldsymbol{s} : R \longrightarrow \mathbb{TFN}, \quad \boldsymbol{s} = \sum_{\ell=1}^{M} \alpha_\ell B_\ell, \ \alpha_\ell \in \mathbb{TFN}, \ \ell = 1, \ldots, M\}.$$

Now, we consider the following interpolation problem:
Given $\mathcal{U} = \{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_N\} \subset \mathbb{TFN}$, find a fuzzy function $\boldsymbol{s} \equiv (s_1, s_2, s_3, s_4) \in S_3(T_n \times T_m; \mathbb{TFN})$ such that

$$\begin{cases} \boldsymbol{s}(x_i, y_j) = \boldsymbol{u}_\ell, & \ell = (n+1)j + i + 1, \\ & i = 0, \ldots, n, \ \ j = 0, \ldots, m, \\ L_2^N s_k = 0, & k = 1, 2, 3, 4, \end{cases} \tag{13}$$

Theorem 3 *Problem* (13) *has a unique solution* $\boldsymbol{\sigma} \in S_3(T_n, T_m; \mathbb{TFN})$ *given by*

$$\boldsymbol{\sigma}(x, y) = \sum_{i=1}^{M} \alpha_i B_i(x, y), \ \ (x, y) \in R,$$

where $\boldsymbol{\Lambda} \equiv (\boldsymbol{\alpha_1}, \ldots, \boldsymbol{\alpha_M}) \in \mathbb{TFN}$ *are the solution of the linear systems* $\boldsymbol{A}\boldsymbol{\Lambda} = \boldsymbol{b}$, *where* $\boldsymbol{A} = \begin{pmatrix} \boldsymbol{A}_1 \\ \boldsymbol{A}_2 \end{pmatrix}$ *is given by* (6) *and* $\boldsymbol{b} = \begin{pmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \end{pmatrix}$, *with* $\boldsymbol{b}_1 = (\boldsymbol{u}_\ell)_{\ell=1,\ldots,N}$ *and* $\boldsymbol{b}_2 = (\boldsymbol{0})_{\ell=1,\ldots,M-N}$, *being* $\boldsymbol{0} \equiv (0, 0, 0, 0)$ *and* $\boldsymbol{u}_\ell = (u_{\ell 1}, u_{\ell 2}, u_{\ell 3}, u_{\ell 4})$, $\ \ell = 1, \ldots, N$.

Proof For any $k = 1, 2, 3, 4$, taking into account Theorem 1, there exists a unique $\sigma_k \in S_3(T_n \times T_m)$ such that

$$\begin{cases} \sigma_k(x_i, y_j) = u_{\ell k}, & \ell = (n+1)j + i + 1, \\ & i = 0, \ldots, n, \ \ j = 0, \ldots, m \\ L_2^N \sigma_k = 0, & \end{cases} \tag{14}$$

Let $\boldsymbol{\sigma} : R \longrightarrow \mathbb{R}^4$ given by $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3, \sigma_4)$, then $\boldsymbol{\sigma} \in (S_3(T_n \times T_m))^4$. Also, taking into account Theorem 1 for $\ell = 1, \ldots, N$, there exists a unique $L_\ell \in S_3(T_n \times T_m)$ such that

$$\begin{cases} L_\ell(\boldsymbol{a}_r) = \delta_{\ell,r} = \begin{cases} 1, & \text{if } \ell = r, \\ 0, & \text{otherwise}, \end{cases} & r = 1, \ldots, N, \\ L_2^N(L_\ell) = 0, & \end{cases}$$

being $\boldsymbol{a}_r = (x_i, y_j)$, for any $i = 0, \ldots, n, \ \ j = 0, \ldots, m$, with $r = (n+1)j + i + 1$. Then

$$\sigma_k(x, y) = \sum_{1}^{N} u_{\ell k} \, L_\ell(x, y), \ \ k = 1, 2, 3, 4. \tag{15}$$

From $u_{\ell 1} \leq u_{\ell 2} \leq u_{\ell 3} \leq u_{\ell 4},\ \ell = 1, \ldots, N$ and (15) we obtain that

$$\sigma_1(x, y) \leq \sigma_2(x, y) \leq \sigma_3(x, y) \leq \sigma_4(x, y), \quad (x, y) \in R.$$

Then $\boldsymbol{\sigma}(x, y) \in \mathbb{TFN}$, for any $(x, y) \in R$, and hence $\boldsymbol{\sigma} \in S_3(T_n, T_m; \mathbb{TFN})$. □

6 Convergence result

Let $\boldsymbol{f} : R \longrightarrow \mathbb{TFN}$ be a fuzzy function

$$\boldsymbol{f}(x, y) = (f_1(x, y), f_2(x, y), f_3(x, y), f_4(x, y)), \text{ for all } (x, y) \in R,$$

being $f_i \in \mathcal{C}^4(R), i = 1, 2, 3, 4$. Let $\boldsymbol{\sigma} \in S_3(T_n \times T_m; \mathbb{TFN})$ the fuzzy bicubic spline verifying (13) for $\boldsymbol{u}_\ell = \boldsymbol{f}(\boldsymbol{a}_\ell),\ \ell = 1, \ldots, N$, and $\boldsymbol{a}_\ell = (x_i, y_j),\ i = 0, \ldots, n,\ j = 0, \ldots, m,\ \ell = (n+1)j + i + 1$. Let $h = max \left\{ \dfrac{b-a}{n}, \dfrac{d-c}{m} \right\}$. From (2), we also have

$$h = \mathcal{O}(\frac{1}{N}),\ N \to +\infty \tag{16}$$

Theorem 4 *Suppose hypothesis* (2) *holds. Then, for any* $(x, y) \in R$

$$d(\boldsymbol{\sigma}(x, y), \boldsymbol{f}(x, y)) = \mathcal{O}(h^4),\ N \to +\infty$$

and thus

$$\lim_{N \to +\infty} d(\boldsymbol{\sigma}(x, y), \boldsymbol{f}(x, y)) = 0.$$

Proof From Definition 3, we have

$$d(\boldsymbol{u}, \boldsymbol{v}) = \max_{i=1,2,3,4} |u_i - v_i|, \tag{17}$$

for all $\boldsymbol{u} = (u_1, u_2, u_3, u_4), \boldsymbol{v} = (v_1, v_2, v_3, v_4) \in \mathbb{TFN}$. From Theorem 2 and (2) we obtain that

$$|f_i(x, y) - \sigma_i(x, y)| = \mathcal{O}(h^4),\ N \to +\infty \tag{18}$$

for all $i = 1, 2, 3, 4$ and $(x, y) \in R$. From (17) and (18) we deduce that

$$d(\boldsymbol{\sigma}(x, y), \boldsymbol{f}(x, y)| = \mathcal{O}(h^4),\ N \to +\infty,$$

and thus

$$\lim_{N \to +\infty} d(\boldsymbol{\sigma}(x, y), \boldsymbol{f}(x, y)) = 0,$$

for any $(x, y) \in R$. □

7 Similarity measures of fuzzy numbers

The concept of similarity of fuzzy numbers is fundamental in the field of fuzzy decision making [9], fuzzy risk and safety analysis [18], pipping risk assesment, batch crystallizer, combustion processes, food production, fluidized catalytic cracking units and chemical separation processes in general; see [11,14,16] and references therein.

In this section, we consider some existing similarity measures of fuzzy numbers. If $A = (a_1, a_2, a_3, a_4)$ and $B = (b_1, b_2, b_3, b_4)$, then the degree of similarity $S(A, B)$ between the trapezoidal fuzzy numbers A and B is defined by Chen [6] as follows:

$$S_{CHEN}(A, B) = 1 - \frac{\sum_{i=1}^{4} \mid a_i - b_i \mid}{4} \in [0, 1]$$

where $\mid a \mid$ is the absolute value of the real number a.
In [13] Lee proposed another similarity measure as follows :

$$S(A, B) = 1 - \frac{\parallel A - B \parallel_{l_p}}{\parallel U \parallel} \times 4^{-\frac{1}{p}},$$

where U is the universe of discourse (the range of all possible values for an input to a fuzzy variable or system)

$$\parallel A - B \parallel_{l_p} = \left(\sum_{i=1}^{4} |a_i - b_i|^p \right)^{\frac{1}{p}}.$$

and $\parallel U \parallel = max(U) - min(U)$.
Hsieh et al. [8] proposed a similarity measure using the *graded mean integration-representation distance* where the degree of similarity $S(A, B)$ between the fuzzy numbers A and B is calculated as follows:

$$S_{HSIEH}(A, B) = \frac{1}{1 + d(A, B)},$$

where $d(A, B) = \mid P(A) - P(B) \mid$, and $P(A),\ P(B)$ are the graded mean integration representations of A and B, respectively. If A and B are trapezoidal fuzzy numbers, with $A = (a_1, a_2, a_3, a_4)$ and $B = (b_1, b_2, b_3, b_4)$, then the graded mean integration of these fuzzy numbers is defined as:

$$P(A) = \frac{a_1 + 2a_2 + 2a_3 + a_4}{6},$$
$$P(B) = \frac{b_1 + 2b_2 + 2b_3 + b_4}{6}.$$

Chen and Chen [7] presented another similarity measure between generalized trapezoidal fuzzy numbers. They presented the (simple center of gravity method) denoted as SCGM to calculate the center of gravity points (x_A^*, y_A^*) and (x_B^*, y_B^*) of the generalized trapezoidal fuzzy number A and B respectively. $A = (a_1, a_2, a_3, a_4)$, $0 \leq a_1 \leq a_2 \leq a_3 \leq a_4 \leq 1$, and $B = (b_1, b_2, b_3, b_4)$, $0 \leq b_1 \leq b_2 \leq b_3 \leq b_4 \leq 1$. Then the degree of similarity $S(A, B)$ between the trapezoidal fuzzy numbers A and B, using the SCGM methodology, is calculated as follows :

$$S_{SCGM}(A, B) = 1 - \frac{\sum_{i=1}^{4} \mid a_i - b_i \mid}{4} \left(1 - \mid x_A^* - x_B^* \mid\right)^{B(S_A, S_B)} \frac{min(y_A^*, y_B^*)}{max(y_A^*, y_B^*)},$$

where $S(A, B) \in [0, 1]$, and

$$x_A^* = \frac{y_A^*(a_3 + a_2) + (a_4 + a_1)(1 - y_A^*)}{2},$$

$$y_A^* = \begin{cases} \frac{1}{2}, & if \quad a_1 = a_4, \\ \frac{1}{6}(\frac{a_3 - a_2}{a_4 - a_1} + 2), & if \quad a_1 \neq a_4, \end{cases}$$

and

$$B(S_A, S_B) = \begin{cases} 1, & if \quad S_A + S_B > 0, \\ 0, & if \quad S_A + S_B = 0, \end{cases}$$

where S_A and S_B are the lengths of the bases of trapezoidal fuzzy numbers A and B, respectively, and defined by:

$$S_A = a_4 - a_1,$$
$$S_B = b_4 - b_1.$$

8 Simulation results

In this section, different interpolation error and similarity estimations are proposed in order to analyze the presented fuzzy interpolation method. The definition of these estimations is

$$\overline{S} = \frac{1}{Z} \sum_{i=1}^{Z} S(f(\xi_i), \sigma(\xi_i)), \tag{19}$$

where $\{\xi_1, \ldots, \xi_Z\} \subset R$ is a set of Z random points in the domain R and S is the Chen (S_{CHEN}) index, the Hsieh index (S_{HSIEH}), the Chen and Chen index (S_{SCGM}), defined in Sect. 7, or the Haussdorf distance (S_d) given in Definition 3. From Theorem 4, it should be verified that $\overline{S}$ tends to 1 as $N \to +\infty$, for $S = S_{CHEN}, S_{HSIEH}, S_{SCGM}$, and $\overline{S}$ tends to 0 as $N \to +\infty$, for $S = S_d$.

To test our method we consider two examples for two fuzzy functions and for different partitions of its domains.

Example 1 $f_1 : [0, \pi] \times [0, \pi] \longrightarrow \mathbb{TFN}$,

$$f_1(x, y) =$$
$$(0.5 \sin(\tfrac{\pi}{2} - x)^2 \cos(\tfrac{\pi}{2} - y)^2 - 0.2 \sin(\tfrac{\pi}{2} - x)^2,$$
$$0.5 \sin(\tfrac{\pi}{2} - x)^2 \cos(\tfrac{\pi}{2} - y)^2 + 0.2 \sin(\tfrac{\pi}{2} - x)^2 + 0.2,$$
$$0.4 \sin(\tfrac{\pi}{2} - x)^2 \cos(\tfrac{\pi}{2} - y)^2 + 0.3 \sin(\tfrac{\pi}{2} - x)^2 + 0.4,$$
$$-0.2 \sin(\tfrac{\pi}{2} - x)^2 \cos(\tfrac{\pi}{2} - y)^2 + 0.5 \sin(\tfrac{\pi}{2} - x)^2 + 0.5).$$

Example 2 $f_2 : [0, 4\pi] \times [0, 4\pi] \longrightarrow \mathbb{TFN}$,

$$f_2(x, y) =$$
$$(0.01(x - 2\pi)^2 + 0.4 \sin(0.2(x - 2\pi)^2)e^{-0.1(y-2\pi)^2} - 0.04,$$
$$0.01(x - 2\pi)^2 + 0.2 \cos(0.1(y - 2\pi)^2)e^{-0.3(x-2\pi)^2} + 0.5,$$
$$0.01(x - 2\pi)^2 + 0.3 \sin(0.2(x - 2\pi)^2)e^{-0.05(y-2\pi)^2} + 1,$$
$$0.01(x - 2\pi)^2 + 0.3 \cos(0.2(x - 2\pi)^2)e^{-0.05(y-2\pi)^2} + 1.6).$$

For the simulations presented in this section, the number of points to compute the estimation $\overline{S}$ given by (19) is $Z = 500$ in all cases.

Figures 2 and 3 show the graph of the fuzzy functions f_1 and f_2 and its fuzzy interpolating bicubic splines for $n = m = 4$ in the first case, and $n = m = 6$ in the second case. The error estimations are $S_d = 7.8692 \times 10^{-2}$ and $S_d = 1.0454 \times 10^{-1}$ respectively.

Tables 1 and 2 illustrate, for Examples 1 and 2, the performance of the fuzzy interpolating bicubic spline for different values of the knot numbers n and m, taking $n = m$ in all cases. The proposed simulations show the influence and relative importance of the knot number in the effectiveness of the approximation. Specifically, the error estimation $\overline{S}_d$ decreases to 0 as $N = (n + 1) \times (m + 1)$ tends to $+\infty$ and the similarity index estimations increases to 1 as $N \to +\infty$.

9 Conclusions

Fuzzy theory based methods (see for example [4]) can help to solve, among many others, the following problems of contemporary analytical chemistry:

- handling uncertain and incomplete data sets or identifying blurred spectra in infrared/ultraviolet spectroscopy or in chromatography,
- modelling data in cases where the assumed model is not exactly valid,
- incorporating and managing, in a rigorous and consistent way, uncertain, inconsistent and/or incomplete information in modern automatic and expert analytical systems.

In this way, by using the fuzzy approach, the error in observations is modelled by the concept of membership to the set of possible and predicted concentration values by means of an appropriate membership function, without the necessity of introducing probability based assumptions.

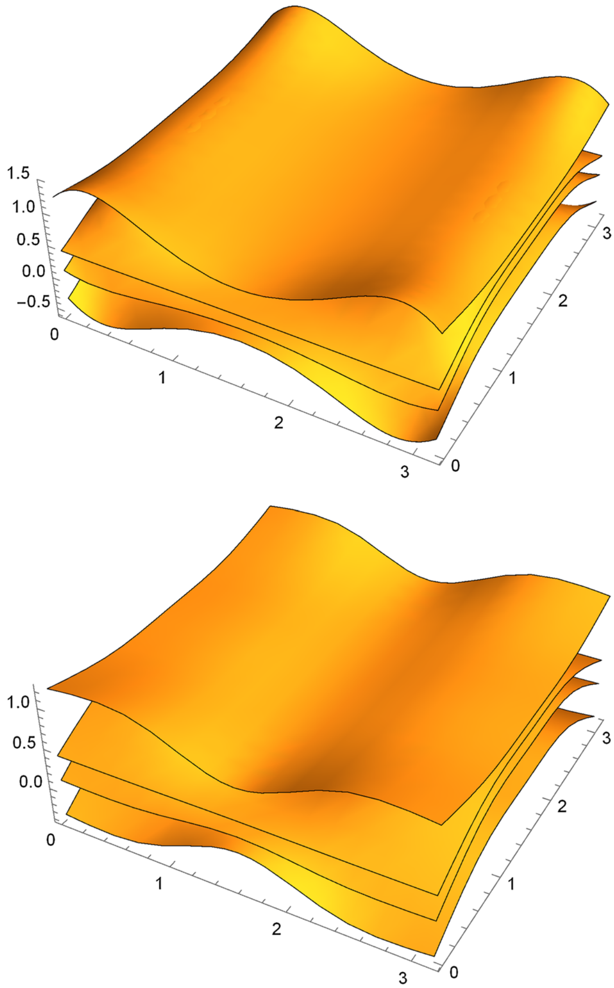

Fig. 2 Example 1. From top to bottom, graph of the fuzzy function $\boldsymbol{f}_1$ and its fuzzy interpolating bicubic spline from a partition of the domain in 4×4 equal squares ($n = m = 4$). The error estimation is $\overline{S}_d = 7.8692 \times 10^{-2}$

Also, reviewing references concerning the interpolation and approximation of fuzzy data, there is a significant lack of development of an interpolation method for a 3D fuzzy data set or a fuzzy bivariate function.

In this paper, we present a fuzzy interpolation method of 3D fuzzy data or fuzzy bivariate functions. We study the solution of this problem and we establish a convergence result about the presented method.

Then, we develop a similar methodology as in [19] to define and use error and similarity indices suitable for the 3D interpolation problem of fuzzy data by means of fuzzy bivariate spline functions.

Two different examples with two-variables fuzzy functions have been presented in order to analyze the behavior of these indices.

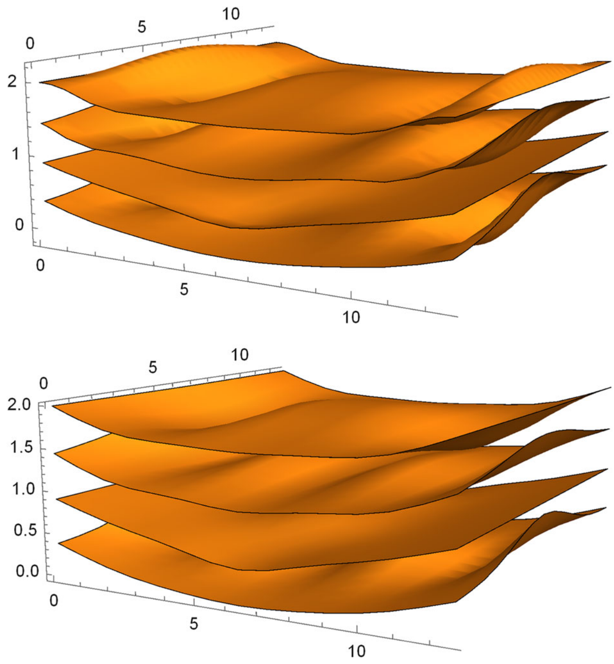

Fig. 3 Example 2. From top to bottom, graph of the fuzzy function f_2 and its fuzzy interpolating bicubic spline from a partition of the domain in 6×6 equal squares ($n = m = 6$). The error estimation is $\overline{S}_d = 1.0454 \times 10^{-1}$

Table 1 Example 1. Error and similarity indices estimates for different knot numbers

Values of $n = m$	$\overline{S}_d$	$\overline{S}_{CHEN}$	$\overline{S}_{HSIEH}$	$\overline{S}_{SCGM}$
4	7.8692×10^{-2}	0.949328	0.955893	0.955190
6	2.7178×10^{-2}	0.981314	0.982188	0.982048
8	9.8117×10^{-3}	0.992527	0.993364	0.993119
10	5.4005×10^{-3}	0.995724	0.996427	0.996109
12	3.0842×10^{-3}	0.997768	0.997906	0.997895
20	6.9539×10^{-4}	0.999533	0.999617	0.999539

Table 2 Example 2. Error and similarity indices estimates for different knot numbers

Values of $n = m$	$\overline{S}_d$	$\overline{S}_{CHEN}$	$\overline{S}_{HSIEH}$	$\overline{S}_{SCGM}$
4	1.5776×10^{-1}	0.926303	0.945866	0.942131
6	1.0454×10^{-1}	0.950933	0.956212	0.954863
8	5.4174×10^{-2}	0.979548	0.979488	0.981600
10	2.5779×10^{-2}	0.988891	0.987013	0.987599
12	1.6717×10^{-2}	0.989741	0.991822	0.992621
20	2.3462×10^{-3}	0.998460	0.998929	0.998613

Analyzing the results presented in Sect. 8 it can be concluded that the proposed error and similarity indices estimations confirm the effectiveness of this method and the convenience of using it in all kind of chemical and other similar engineering situations.

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