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A new error bound for nonzero initial conditions in linear dynamical systems

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Balanced truncation is one of the most widely used techniques for reducing the order of asymptotically stable linear time-invariant systems because it preserves stability and provides computable *a priori* error bounds. However, the classical theory is formulated essentially for homogeneous initial conditions and therefore does not directly quantify the approximation error when the system starts from a nonzero state. In many applications, such as electrical circuits with pre-charged capacitors, mechanical systems released from displaced configurations, and diffusion processes initialized away from equilibrium, the initial condition contributes substantially to the output and must be accounted for in model reduction.

In this paper we derive a new L_2-error bound for balanced truncation of continuous-time linear time-invariant systems with nonzero initial conditions. The key idea is to reinterpret the initial state as an auxiliary impulsive input acting through an additional channel of an extended realization. This reformulation preserves the input–output behavior of the original initial-value problem while allowing the use of the standard balanced-truncation framework. We show that the observability Gramian remains unchanged, whereas the controllability Gramian acquires a transparent additive correction induced by the initial condition. Based on this structure, we derive an explicit *a priori* error estimate in which the contribution of the initial state and the contribution of the external input are separated. The resulting bound is independent of regularization parameters and reduces to the classical balanced-truncation bound in the homogeneous case.

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Numerical experiments for a SLICOT benchmark model, an RC-circuit example, and an additional illustrative test problem show that the proposed estimate is reliable and substantially sharper than existing bounds for systems with inhomogeneous initial conditions.

Keywords: Balanced truncation; model order reduction; linear time-invariant systems; nonzero initial conditions; inhomogeneous initial conditions; L_2-error bound; Hankel singular values; observability Gramian; Dirac delta input.

1. Introduction

Model order reduction has become a standard tool in the analysis, simulation, optimization, and control of large-scale dynamical systems arising in mechanics, circuit theory, fluid dynamics, thermal processes, and many other fields. Spatial discretization of partial differential equations, network interconnections, and multiphysics couplings often produce state-space systems with hundreds, thousands, or even millions of degrees of freedom. In such settings, direct simulation or controller design based on the full-order model may be prohibitively expensive, and one therefore seeks a reduced-order surrogate that accurately reproduces the relevant input–output behavior at much lower computational cost.[1–6]

Among projection-based methods, balanced truncation (BT) occupies a central position. It is attractive because it preserves asymptotic stability for stable systems, it is based on physically meaningful energy concepts through controllability and observability Gramians, and it provides a computable *a priori* error estimate in terms of the neglected Hankel singular values.[1–3,7,8] For a stable continuous-time linear time-invariant (LTI) system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t), \tag{1}$$

the classical BT theory yields, under the homogeneous initial condition $x(t_0) = 0$, the well-known estimate

$$\|y - \bar{y}\|_{L_2(t_0,\infty)} \leq 2 \sum_{i=r+1}^{n} \sigma_i \, \|u\|_{L_2(t_0,\infty)}, \tag{2}$$

where $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$ are the Hankel singular values and $\bar{y}$ denotes the output of the reduced model of order r.[2,3,6,7]

Despite its success, the standard BT bound is incomplete in the presence of nonzero initial conditions. In many practical systems the initial state is not negligible. For example, an RC circuit may start with charged capacitors, so even in the absence of an external driving signal the stored electric energy generates a transient output. Likewise, a mechanical structure may be released from a displaced configuration, producing a free response that is entirely driven by the initial condition rather than by the input. In diffusion-reaction systems and thermal models, nonuniform initial profiles may dominate the short-time dynamics. In all these cases, a reduced model should approximate not only the forced response but also the free response associated with the initial state.

This observation has motivated several works on model reduction with inhomogeneous or nonzero initial conditions. A rigorous and influential extension of balanced truncation was proposed by Heinkenschloss *et al.*,[9] who augmented the reduction framework to account for the initial-value contribution and derived corresponding error estimates. Related developments include singular perturbation approximation for nonzero initial conditions,[10,11] Arnoldi-type approaches,[12] bilinear extensions,[13] port-Hamiltonian settings,[14] and second-order or delay-system formulations with inhomogeneous initial states.[15,16] More recently, Schröder and Voigt[17] proposed balanced-truncation procedures with *a priori* error bounds for LTI systems with nonzero initial value, providing a modern reference point for the field.

Main contribution. The main contribution of this paper is the derivation of a new *explicit* L_2-error bound for balanced truncation of stable continuous-time LTI systems with nonzero initial conditions. The bound has three distinguishing features:

(1) it separates the effect of the initial condition from the effect of the external input;
(2) it does not depend on any regularization parameter;
(3) It is expressed in terms of quantities already available in the standard BT pipeline, namely the observability Gramian, the balancing transformation, and the Hankel singular values.

Conceptual novelty versus Heinkenschloss–Reis–Antoulas. The central idea of our approach is to embed the initial state as an auxiliary impulsive input through an additional channel of an extended system. At first sight this may appear close in spirit to the framework of Heinkenschloss *et al.*,[9] but this analysis differs in two essential respects. First, we use the extended realization to derive a bound in which the initial-condition contribution is written directly through Gramian-based Euclidean norms, rather than through regularized input surrogates and auxiliary optimization ingredients. Second, our final estimate retains the classical BT tail term for the external input unchanged, while the initial-condition effect appears as a separate additive quantity. This clearer separation provides a more transparent interpretation of the reduction error and is easier to evaluate in practical computations.

The paper is organized as follows. Section 2 reviews the relevant background on LTI systems, Gramians, and balanced truncation. Section 3 discusses balanced truncation with nonzero initial conditions. Section 4 introduces the extended realization, analyzes its Gramians, and derives the new L_2 error bound. Section 5 presents three numerical examples. Section 6 concludes the paper and briefly comments on possible extensions to discrete-time and time-varying settings.

2. Preliminaries and Problem Formulation

We consider asymptotically stable continuous-time LTI systems of the form

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t), \quad x(t_0) = X_0, \tag{3}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, and all eigenvalues of A have a strictly negative real part. The state, input, and output are denoted by $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and $y(t) \in \mathbb{R}^p$, respectively.

The solution is given by

$$x(t) = e^{A(t-t_0)}X_0 + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\,d\tau, \quad y(t) = Cx(t). \tag{4}$$

The first term is the free response generated by the initial state X_0, while the second term is the forced response generated by the input u.

2.1. *Controllability and observability Gramians*

For stable systems, the controllability and observability Gramians are defined by

$$W_c = \int_0^\infty e^{At}BB^Te^{A^Tt}\,dt, \tag{5}$$

$$W_o = \int_0^\infty e^{A^Tt}C^TCe^{At}\,dt. \tag{6}$$

They solve the Lyapunov equations

$$AW_c + W_cA^T = -BB^T, \tag{7}$$

$$A^TW_o + W_oA = -C^TC. \tag{8}$$

The Gramian W_c measures how strongly each state direction can be excited by the input, while W_o measures how strongly each state direction is seen at the output.[1–3,8]

2.2. *Balanced truncation in the homogeneous case*

Balanced truncation seeks a coordinate transformation $x = T\hat{x}$ such that, in the transformed variables,

$$\hat{W}_c = \hat{W}_o = \Sigma = \mathrm{diag}(\sigma_1, \dots, \sigma_n), \tag{9}$$

where $\sigma_1 \geq \cdots \geq \sigma_n \geq 0$ are the Hankel singular values. Partitioning

$$\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}, \quad \Sigma_1 = \mathrm{diag}(\sigma_1, \dots, \sigma_r)$$

and truncating the states corresponding to Σ_2 yields a reduced model of order r. For zero initial condition, the classical BT bound is exactly (2).

2.3. *Why nonzero initial conditions matter*

When $X_0 \neq 0$, the free response

$$y_{\text{free}}(t) = Ce^{A(t-t_0)}X_0,$$

contributes to the total output even if $u \equiv 0$. Hence, the reduced model must approximate two distinct mechanisms:

- the input-driven dynamics;
- the transient dynamics caused by the initial state.

This is the setting addressed in this paper.

3. Balanced Truncation with Nonzero Initial Conditions

The main idea is to reinterpret the initial condition as an impulsive auxiliary input. This allows us to recast the initial-value problem into a zero-initial-condition problem for an extended system.

3.1. *Extended realization*

Consider the system

$$\dot{\tilde{x}}(t) = A\tilde{x}(t) + [B \quad X_0]\begin{bmatrix} u(t) \\ \delta_0(t) \end{bmatrix}, \quad \tilde{y}(t) = C\tilde{x}(t), \quad \tilde{x}(t_0) = 0, \tag{10}$$

where δ_0 denotes the Dirac delta at the initial time t_0. Using variation of constants,

$$\begin{aligned} \tilde{x}(t) &= \int_{t_0}^{t} e^{A(t-\tau)}[B \quad X_0]\begin{bmatrix} u(\tau) \\ \delta_0(\tau) \end{bmatrix} d\tau \\ &= \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\, d\tau + \int_{t_0}^{t} e^{A(t-\tau)}X_0\delta_0(\tau)\, d\tau \\ &= \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\, d\tau + e^{A(t-t_0)}X_0. \end{aligned} \tag{11}$$

Hence,

$$\tilde{x}(t) = x(t), \quad \tilde{y}(t) = y(t), \quad t \geq t_0. \tag{12}$$

Thus the extended system is exactly equivalent, in input–output terms, to the original initial-value problem.

3.2. *Effect on the Gramians*

The observability Gramian remains unchanged because the output matrix is unchanged:

$$\tilde{W}_o = W_o. \tag{13}$$

By contrast, the controllability Gramian of the extended system becomes

$$\begin{aligned}\tilde{W}_c &= \int_0^\infty e^{At}(BB^T + X_0X_0^T)e^{A^Tt}\,dt \\ &= W_c + \int_0^\infty e^{At}X_0X_0^Te^{A^Tt}\,dt. \end{aligned} \tag{14}$$

Therefore, the initial condition contributes an additional positive semidefinite correction to the controllability Gramian.

3.3. *Choice of the factor L*

A referee correctly observed that the factorization

$$W_o = L^T L$$

is not unique. In this work, we adopt the canonical symmetric choice based on the eigendecomposition of the observability Gramian,

$$W_o = Q\Lambda Q^T, \quad L = \Lambda^{1/2}Q^T, \tag{15}$$

where Q is orthogonal and $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ with $\lambda_i \geq 0$. This choice is numerically convenient and makes the quantity

$$\|LX_0\|_2 = \sqrt{X_0^T W_o X_0},$$

independent of the particular factorization. In fact, if $\widetilde{L}^T\widetilde{L} = W_o$, then

$$\|\widetilde{L}X_0\|_2^2 = X_0^T W_o X_0 = \|LX_0\|_2^2,$$

so the value entering the bound is invariant under the choice of factor.

4. Extended Realization and New L_2 Error Bound

We now derive the new error estimate.

4.1. *Balanced coordinates*

Let T be a balancing transformation for the original system, and write

$$\hat{x} = T^{-1}x, \quad \hat{X}_0 = T^{-1}X_0 = \begin{bmatrix} \bar{X}_0 \\ X_{0,2} \end{bmatrix},$$

where $\bar{X}_0 \in \mathbb{R}^r$ denotes the retained part of the initial state in balanced coordinates.

4.2. *Initial-condition contribution*

For $u \equiv 0$, the output of the full model is

$$y_0(t) = Ce^{A(t-t_0)}X_0.$$

Using the observability Gramian,

$$\begin{aligned}\|y_0\|^2_{L_2(t_0,\infty)} &= \int_{t_0}^{\infty} X_0^T e^{A^T(t-t_0)} C^T C e^{A(t-t_0)} X_0 \, dt \\ &= X_0^T W_o X_0 = \|LX_0\|_2^2. \end{aligned} \tag{16}$$

Hence,

$$\|y_0\|_{L_2(t_0,\infty)} = \|LX_0\|_2. \tag{17}$$

Similarly, in balanced coordinates the reduced free response satisfies

$$\|\bar{y}_0\|_{L_2(t_0,\infty)} = \|\Sigma_1^{1/2} \bar{X}_0\|_2. \tag{18}$$

Therefore, by the triangle inequality,

$$\|y_0 - \bar{y}_0\|_{L_2(t_0,\infty)} \leq \|LX_0\|_2 + \|\Sigma_1^{1/2} \bar{X}_0\|_2. \tag{19}$$

4.3. *Input contribution*

For $X_0 = 0$, the classical BT estimate gives

$$\|y_u - \bar{y}_u\|_{L_2(t_0,\infty)} \leq 2 \sum_{i=r+1}^{n} \sigma_i \, \|u\|_{L_2(t_0,\infty)}. \tag{20}$$

4.4. *Main theorem*

Combining (19) and (20) yields the main result.

Theorem 1. *Let the continuous-time LTI system* (3) *be asymptotically stable, and let T be a balancing transformation such that the balanced Gramians are equal to*

$$\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_n).$$

Let $\Sigma_1 = \mathrm{diag}(\sigma_1, \ldots, \sigma_r)$ correspond to the retained states, and let $\bar{X}_0$ denote the retained part of $T^{-1}X_0$. If $W_o = L^T L$, then for every $u \in L_2(t_0, \infty)$,

$$\|y - \bar{y}\|_{L_2(t_0,\infty)} \leq \|LX_0\|_2 + \|\Sigma_1^{1/2} \bar{X}_0\|_2 + 2 \sum_{i=r+1}^{n} \sigma_i \, \|u\|_{L_2(t_0,\infty)}. \tag{21}$$

The estimate clearly separates the influence of the initial condition from that of the external input. The third term is exactly the classical balanced-truncation tail, while the first two terms capture the initial-state contribution through the observability Gramian and the retained balanced coordinates.

Corollary 1. *If the system is already in balanced coordinates, so that $W_o = \Sigma$, then*

$$\|y - \bar{y}\|_{L_2(t_0,\infty)} \leq \|\Sigma^{1/2} X_0\|_2 + \|\Sigma_1^{1/2} \bar{X}_0\|_2 + 2 \sum_{i=r+1}^{n} \sigma_i \, \|u\|_{L_2(t_0,\infty)}. \tag{22}$$

In particular, using $\|\Sigma_1^{1/2}\bar{X}_0\|_2 \leq \|\Sigma^{1/2}X_0\|_2$, *one obtains the simpler estimate*

$$\|y-\bar{y}\|_{L_2(t_0,\infty)} \leq 2\|\Sigma^{1/2}X_0\|_2 + 2\sum_{i=r+1}^{n} \sigma_i \,\|u\|_{L_2(t_0,\infty)}. \tag{23}$$

Remark 1. The theorem does not require any regularization of the Dirac delta input. The latter serves only as a conceptual device for constructing the equivalent extended realization.

5. Numerical Examples and Results

This section illustrates the behavior of the proposed error bound on three examples. In each case we compare:

(1) The true L_2 output error $\|y-\bar{y}\|_{L_2}$;
(2) The proposed bound (21);
(3) The classical BT tail $2\sum_{i=r+1}^{n}\sigma_i$ when relevant.

5.1. *SLICOT benchmark example*

We first consider a standard benchmark model from the SLICOT collection.[18] The full system has dimension $n = 48$, and we compute reduced models for several values of r. Figures 3–6 display the decay of the Hankel singular values and the corresponding error curves.

Table 1 reports the true output error, the proposed bound, and the classical BT tail term for selected reduced orders. The results show that the proposed estimate

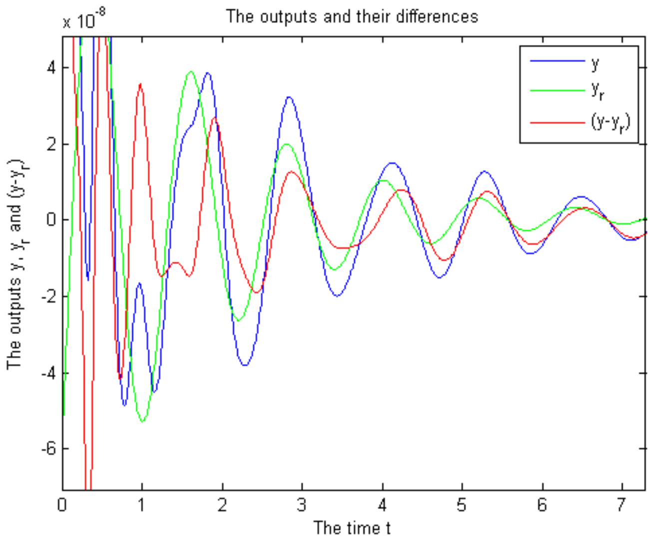

Fig. 1. True L_2 output error for the SLICOT benchmark with $n = 48$ and $r = 3$.

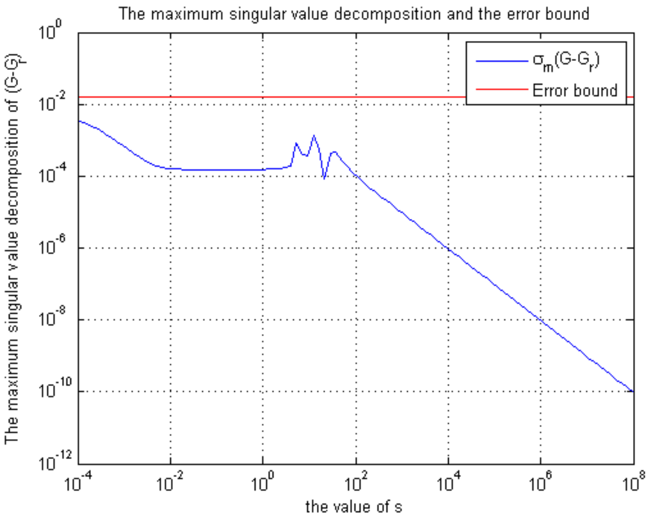

Fig. 2. Proposed error bound for the SLICOT benchmark with $n = 48$ and $r = 3$.

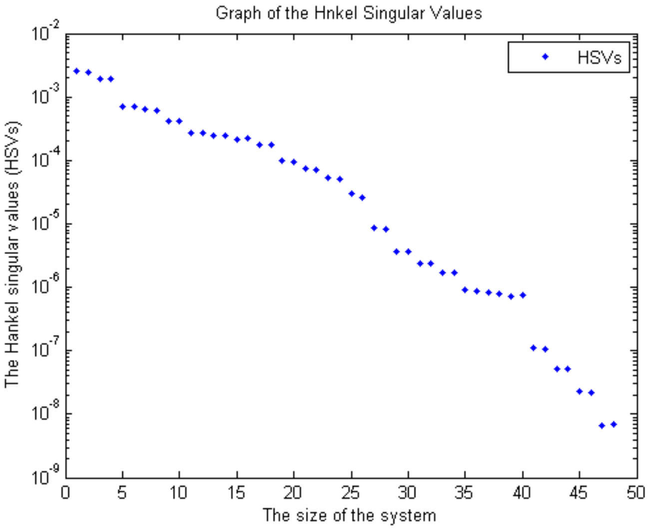

Fig. 3. Neglected Hankel singular values for the SLICOT benchmark with $n = 48$ and $r = 3$.

tracks the true error much more closely than the classical tail term alone, which ignores the effect of the initial state.

5.2. *RC-circuit example*

We next consider an RC-circuit model of order $n = 20$. This example provides a concrete real-world interpretation of nonzero initial conditions: pre-charged capacitors

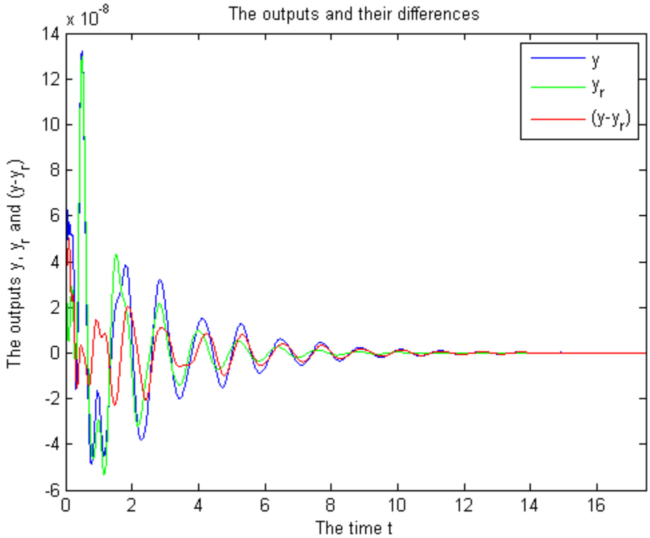

Fig. 4. True L_2 output error for the SLICOT benchmark with $n = 48$ and $r = 6$.

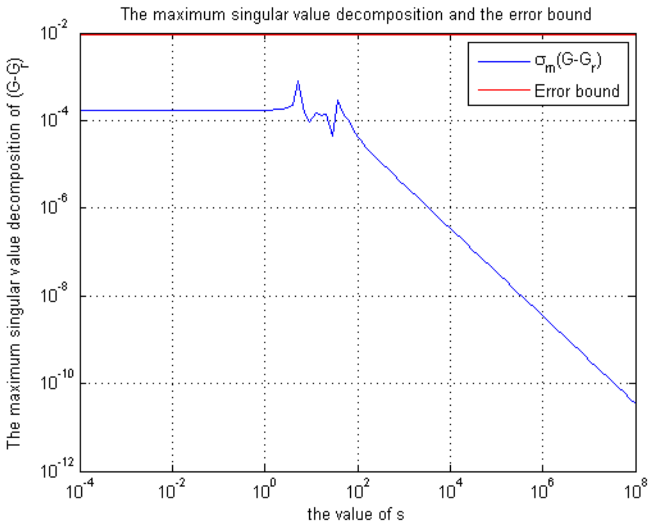

Fig. 5. Proposed error bound for the SLICOT benchmark with $n = 48$ and $r = 6$.

store energy before the external input is applied, and this stored energy produces a transient current or voltage response. Consequently, the initial state is not merely a mathematical artifact but a physically meaningful source of output dynamics.

Figure 7 shows the circuit configuration. Reduced models of orders $r = 3$ and $r = 6$ are computed, and the corresponding numerical results are summarized in Table 2. Again, the proposed bound is much sharper than the classical tail estimate.

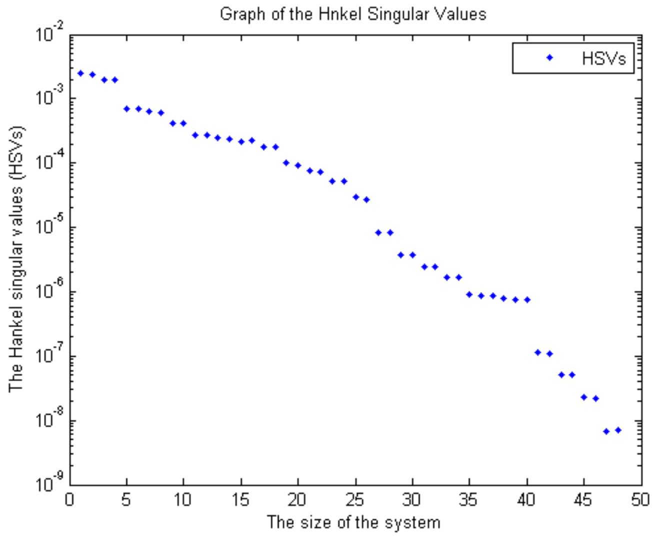

Fig. 6. Neglected Hankel singular values for the SLICOT benchmark with $n = 48$ and $r = 6$.

Table 1. SLICOT benchmark example ($n = 48$): true L_2 error, proposed bound, and classical BT tail.

r	$\|y - \bar{y}\|_{L_2}$	Proposed bound	$2\sum_{i=r+1}^{48} \sigma_i$
1	1.3302×10^{-5}	8.2104×10^{-4}	2.43×10^{-2}
2	1.3322×10^{-5}	7.5478×10^{-4}	1.94×10^{-2}
3	1.3334×10^{-5}	6.9663×10^{-4}	1.56×10^{-2}
4	1.3345×10^{-5}	5.9310×10^{-4}	1.17×10^{-2}
5	1.3349×10^{-5}	5.5003×10^{-4}	1.03×10^{-2}
6	1.3353×10^{-5}	5.2135×10^{-4}	8.90×10^{-3}
7	1.3356×10^{-5}	4.7489×10^{-4}	7.60×10^{-3}
8	1.3359×10^{-5}	4.3172×10^{-4}	6.40×10^{-3}
9	1.3361×10^{-5}	3.9332×10^{-4}	5.50×10^{-3}
10	1.3362×10^{-5}	3.5733×10^{-4}	4.70×10^{-3}
11	1.3364×10^{-5}	3.3283×10^{-4}	4.20×10^{-3}
15	1.3368×10^{-5}	2.3451×10^{-4}	2.20×10^{-3}

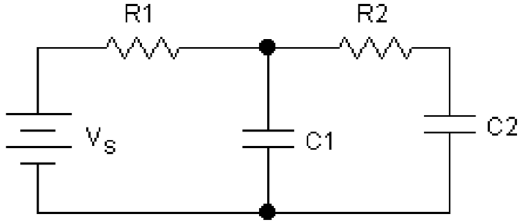

Fig. 7. Simple RC-circuit used as a physical example of an LTI system with nonzero initial condition.

Table 2. RC-circuit example ($n = 20$): true L_2 error, proposed bound, and classical BT tail.

r	$\|y - \bar{y}\|_{L_2}$	Proposed bound	$2\sum_{i=r+1}^{20} \sigma_i$
1	1.3277×10^{-4}	4.38×10^{-2}	1.83367×10^{1}
2	1.3399×10^{-4}	6.90×10^{-3}	7.095×10^{-1}
3	1.3401×10^{-4}	4.50×10^{-3}	3.746×10^{-1}
4	1.3402×10^{-4}	2.30×10^{-3}	8.96×10^{-2}
5	1.3403×10^{-4}	1.50×10^{-3}	4.47×10^{-2}
6	1.3403×10^{-4}	7.2379×10^{-4}	6.50×10^{-3}
7	1.3403×10^{-4}	5.1898×10^{-4}	3.40×10^{-3}
8	1.3403×10^{-4}	3.1239×10^{-4}	7.6225×10^{-4}
9	1.3403×10^{-4}	2.1124×10^{-4}	3.5641×10^{-4}
10	1.3403×10^{-4}	1.2563×10^{-4}	7.8460×10^{-5}
11	1.3403×10^{-4}	8.1746×10^{-5}	3.7626×10^{-5}
15	1.3403×10^{-4}	5.6773×10^{-6}	3.1260×10^{-7}

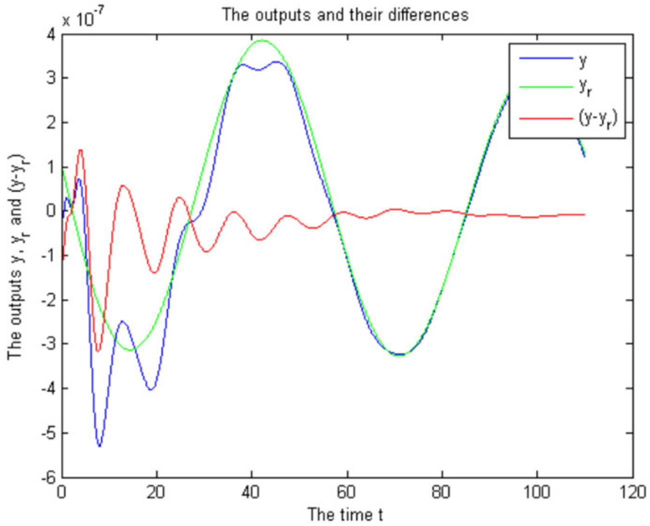

Fig. 8. True L_2 output error for the RC-circuit example with $n = 20$ and $r = 3$.

5.3. *Illustrative small-scale example*

To further illustrate the behavior of the bound, we consider a stable random LTI system of dimension $n = 8$,

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t), \tag{24}$$

with normalized random initial state X_0 and Gaussian input

$$u(t) = e^{-(t-3)^2}, \quad t \in [0, 10]. \tag{25}$$

The computed Hankel singular values are

$$\sigma = (0.602054, 5.10 \times 10^{-3}, 4.21 \times 10^{-4}, 2.42 \times 10^{-5}, 8.32 \times 10^{-7}, 0, 0, 0). \tag{26}$$

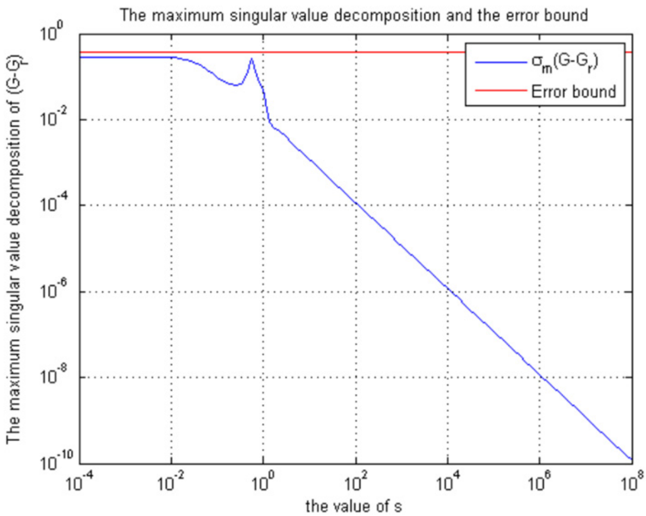

Fig. 9. Proposed error bound for the RC-circuit example with $n = 20$ and $r = 3$.

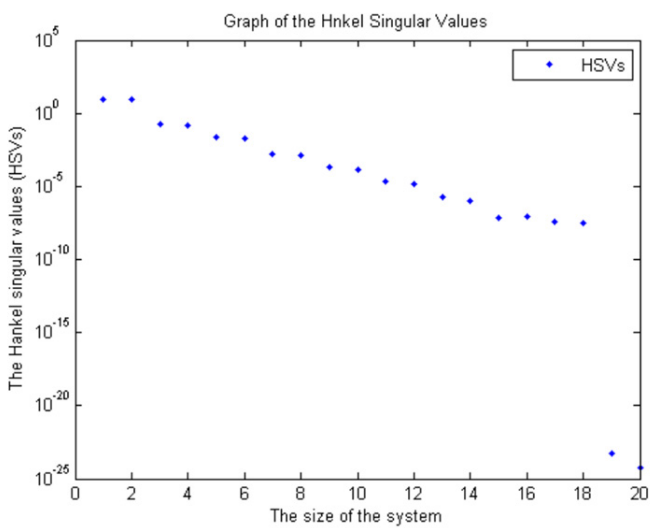

Fig. 10. Neglected Hankel singular values for the RC-circuit example with $n = 20$ and $r = 3$.

Only the first five are numerically nonzero, so the effective balanced dimension is five.

For each reduced order $r = 1, \ldots, 4$, we compute the true output error

$$E_{\text{true}}(r) = \left(\int_0^{10} |y(t) - y_r(t)|^2 \, dt \right)^{1/2} \tag{27}$$

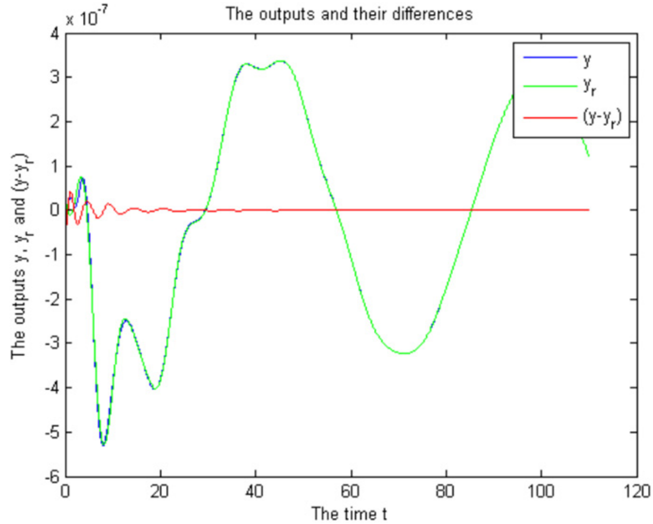

Fig. 11. True L_2 output error for the RC-circuit example with $n = 20$ and $r = 6$.

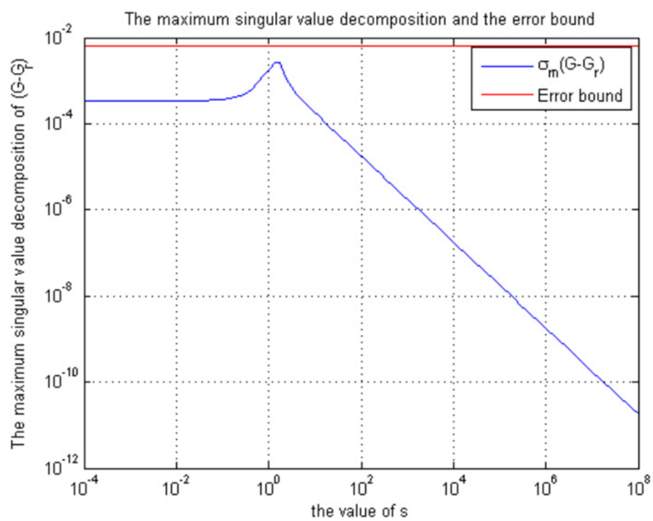

Fig. 12. Proposed error bound for the RC-circuit example with $n = 20$ and $r = 6$.

and the proposed bound

$$E_{\text{new}}(r) = \|LX_0\|_2 + \|\Sigma_r^{1/2} X_{0,r}\|_2 + 2\left(\sum_{i>r} \sigma_i\right) \|u\|_{L^2(0,10)}. \tag{28}$$

The results show that the true error decreases rapidly with the reduced order, while the bound remains valid and mildly nonmonotone. This behavior is fully consistent with the structure of (28): the truncation tail decreases with r, whereas the

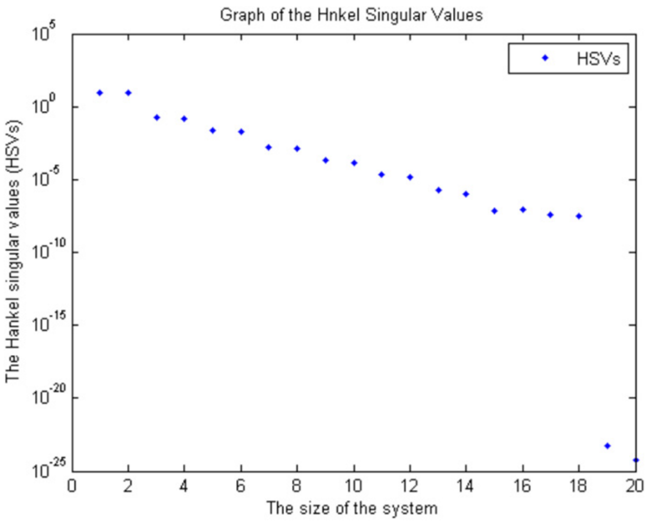

Fig. 13. Neglected Hankel singular values for the RC-circuit example with $n = 20$ and $r = 6$.

Table 3. True output error and proposed bound for the illustrative example.

r	E_{true}	E_{new}
1	0.1393	0.1496
2	0.1013	0.1476
3	0.0427	0.1543
4	0.0046	0.1560

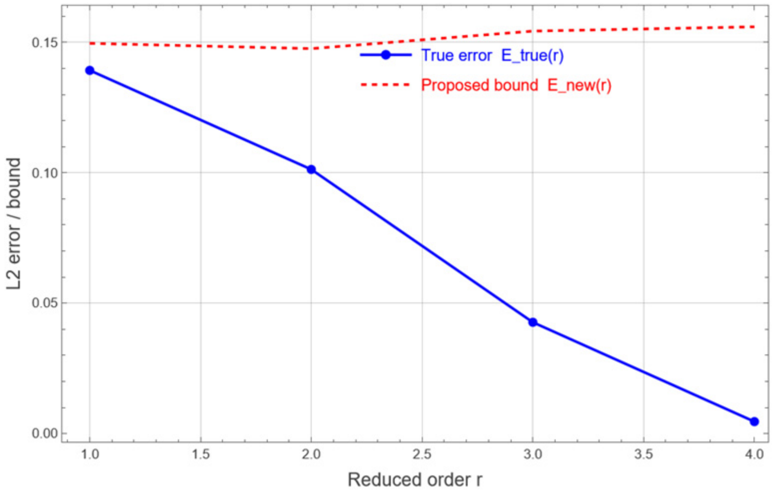

Fig. 14. Comparison of the true L^2 error and the proposed *a priori* bound for the illustrative example.

retained initial-condition contribution may increase as more balanced coordinates are included.

6. Conclusion

We have developed a new *a priori* L_2 error bound for balanced truncation of stable continuous-time LTI systems with nonzero initial conditions. The central idea is to reinterpret the initial state as an auxiliary impulsive input, leading to an extended realization that is exactly equivalent to the original initial-value problem. This perspective yields a transparent analysis of the associated Gramians and leads to an explicit error estimate in which the effect of the initial condition and the effect of the external input are separated.

Compared with earlier bounds for inhomogeneous initial conditions, the present estimate is simpler to interpret, free of regularization parameters, and directly computable from standard BT ingredients. The numerical examples show that it is reliable and significantly sharper than classical bounds that neglect the initial-state contribution.

Although this paper is restricted to continuous-time LTI systems, the underlying idea suggests several natural extensions. For discrete-time systems, one may seek an analogous embedding of the initial state into an augmented input framework adapted to difference equations. For systems with time-varying parameters, a comparable treatment would require time-dependent Gramians and a nonautonomous balancing framework. These directions lie beyond the present scope but constitute promising topics for future work.

Appendix A. Robust Balanced Truncation: Numerical Implementation

This appendix contains the Mathematica implementation used in the illustrative numerical example. The code computes the controllability and observability Gramians, constructs a balanced realization, truncates the balanced model, and evaluates both the true L^2 output error and the proposed *a priori* bound for increasing reduced dimensions.

Important note. Your original Mathematica appendix can be retained here with only two edits:

(1) Replace the bound formula everywhere by the corrected version

$$E_{\text{new}}(r) = \|Lx_0\|_2 + \|\Sigma_r^{1/2} x_0^{(r)}\|_2 + 2 \sum_{i>r} \sigma_i \, \|u\|_{L^2},$$

instead of the version with squared norms;

(2) If you keep the descriptive text of the appendix, make it consistent with the corrected theorem and equation numbers from the main body.

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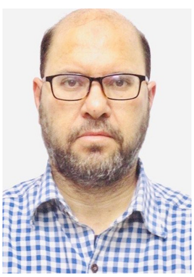

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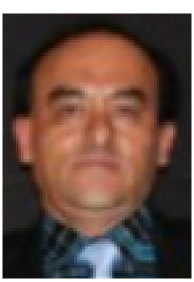

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