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A Confined N-Dimensional Harmonic Oscillator

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Abstract We compute the energy eigenvalues for the N-dimensional harmonic oscillator confined in an impenetrable spherical cavity. The results show their dependence on the size of the cavity and the space dimension N. The obtained results are compared with those for the free N-dimensional harmonic oscillator, and as a result, the notion of fractional dimensions is pointed out. Finally, we examine the correlation between eigenenergies for confined oscillators in different dimensions.

Keywords Foundations · Confined systems · Higher dimensions

1 Introduction

The simple harmonic oscillator is one of the most important topics in quantum mechanics. It is applicable in different physical situations [16] and has the great advantage that it has closed solutions for the energy eigenvalues and eigenfunctions. Various investigations of the harmonic oscillator have been considered. For example, the time- dependent harmonic oscillator [8, 14, 17], harmonic oscillator with delta–function potential [6, 19], an harmonic oscillator [23], and the spiked harmonic oscillator [9, 10]. Recently, there has been some renewed interest in the confined quantum systems [1, 3, 7, 26, 27]. This interest is partly motivated by the rapid technological advances, for example in the field of semiconductor quantum dots [12], where the computation of the electronic structure of such systems necessarily has to take into account the presence of the finite confining boundaries and their effect on the system. During the past years, the generalization of three- dimensional quantum problems to higher dimensions received a considerable development in theoretical and mathematical physics. For example, the N-dimensional analogy of the hydrogen atom has been studied extensively over the years [2, 4, 13, 18]. In addition, the generalization to higher dimensions is useful in Ising limit of quantum field theory, in random walks and in Casimir

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effect [5]. Just recently, the stark effect in hyper-spherical coordinates [20] and closed forms of the Green's function [24] in N-dimensional space have been discussed.

In this paper, we consider an N-dimensional harmonic oscillator that is confined in an impenetrable spherical cavity.

The paper is organized as follows: In Sect. 2, we present the solution to the harmonic oscillator in a spatial dimension N. In Sect. 3, the connection between a confined harmonic oscillator and a free one is analyzed. Section 4 deals with numerical computations of the energy eigenvalues for the confined harmonic oscillator. Section 5 examines the mapping between energy eigenvalues of confined harmonic oscillators in different dimensions. Finally, our conclusions are presented in Sect. 6.

2 The N-Dimensional Harmonic Oscillator

We consider a harmonic oscillator that is confined at the center of an N-dimensional spherical cavity of radius S. It is assumed that the walls of the cavity to be impenetrable. The potential to be considered is

$$V(r) = \begin{cases} \frac{1}{2}kr^2, & r < S, \\ \infty, & r > S. \end{cases} \tag{1}$$

The time-independent Schrödinger equation for the harmonic oscillator is:

$$H\psi(\vec{r}) = \left[\frac{-\hbar^2}{2m}\nabla^2 + \frac{1}{2}kr^2\right]\psi(\vec{r}) = E\psi(\vec{r}), \tag{2}$$

where ∇^2 is the N-dimensional Laplace operator given by [28],

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{N-1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\Omega^2, \tag{3}$$

where r is the radial coordinate and Ω^2 is the Laplace operator on the unit hypersphere S^{N-1}. The eigenfunction $\psi(\vec{r})$ satisfies the Schrödinger equation for $r < S$ and should be regular at the origin. The only difference from the free space—case is that now the wave function must vanish at $r = s$ instead of at $r = \infty$. The Schrödinger equation in (2) splits into radial and angular differential equations with the latter having the hyper-spherical harmonics as solutions (eigenfunctions of Ω^2), $Y_\ell^m(\{\theta_i\})$, with $i = 1, 2, \ldots, N-1$. The radial part solution, $R(r)$, satisfies

$$\left[\frac{-\hbar^2}{2m}\left(\frac{d^2}{dr^2} + \frac{N-1}{r}\frac{d}{dr}\right) + \frac{\ell(\ell+N-1)}{2mr^2} + \frac{1}{2}m\omega^2 r^2\right]R(r) = ER(r), \tag{4}$$

where $\omega^2 = k/m$. By defining $\alpha = (m\omega/\hbar)^{1/2}$, $\lambda = 2E/\hbar\omega$, and $\rho = \alpha r$, and letting $R(r) = u(r)/r^{(N-1)/2}$, (4) becomes

$$\left[\frac{d^2}{dr^2} + \frac{(\ell + \frac{N-3}{2})(\ell + \frac{N-1}{2})}{\rho^2} + \lambda - \rho^2\right]u(\rho) = 0. \tag{5}$$

Looking for solutions of the form

$$u(\rho) = \rho^{\ell + \frac{N-1}{2}} e^{-\rho 2/2} V(\rho), \tag{6}$$

and letting $z=\rho^2$, (5) reduces to

$$z\frac{d^2V}{dz^2}+\left[\left(\ell+\frac{N}{2}\right)-z\right]\frac{dV}{dz}-\left[\frac{1}{2}\left(\ell+\frac{N}{2}\right)-\frac{\lambda}{4}\right]V(z)=0. \tag{7}$$

This can be identified as the Kummer-Laplace differential equation

$$z\frac{d^2V}{dz^2}+(b-z)\frac{dV}{dz}-aV=0, \tag{8}$$

whose solution is the confluent hypergeometric function, ${}_1F_1(a,b,z)$, with the identification

$$a=\frac{1}{2}\left(\ell+\frac{N}{2}\right)-\frac{\lambda}{4}, \tag{9}$$

and

$$b=\ell+\frac{N}{2}. \tag{10}$$

Thus the solution of (7) is

$$V(z)=A\,{}_1F_1\left[\frac{1}{2}\left(\ell+\frac{N}{2}\right)-\frac{\lambda}{4},\ell+\frac{N}{2},z\right], \tag{11}$$

where A is a normalization constant. Therefore, with the help of (6), we have

$$R(r)=Ar^{\ell}\bar{e}^{\alpha^2r2/2}{}_1F_1\left[\frac{1}{2}\left(\ell+\frac{N}{2}\right)-\frac{\lambda}{4},\ell+\frac{N}{2},\alpha^2r^2\right]. \tag{12}$$

To avoid divergence of $R(r)$ as $r\to\infty$, for the free space case, the confluent hypergeometric series must be transformed into a polynomial of degree n_r, by requiring that

$$\frac{1}{2}\left(\ell+\frac{N}{2}\right)-\frac{\lambda}{4}=-n_r,\quad n_r=0,1,2,\ldots. \tag{13}$$

This condition may also be written as

$$\lambda=2\left(n+\frac{N}{2}\right), \tag{14}$$

where

$$n=2n_r+\ell,\quad n=0,1,2,\ldots, \tag{15}$$

and thus,

$$E=\hbar\omega\left(n+\frac{N}{2}\right). \tag{16}$$

Now, we are in a position to consider the effect of the boundary of the cavity on the system. The confluent hypergeometric function

$${}_1F_1(a,b,z)=\sum\frac{(a)_n}{(b)_n}\frac{z^n}{n!}, \tag{17}$$

where the Pochhammer symbol $(a)_n$ is defined by [22]

$$(a)_n = a\,(a+1)\,(a+2)\cdots(a+n-1)\,. \tag{18}$$

The vanishing of ${}_1F_1$ at the radius, S of the cavity, requires a to be negative (since b is positive in our case, see (10)).

The value of a at which ${}_1F_1$ vanishes at $r = S$ determines λ and thus the energy.

If $a = -j$, with $j =$ integer, then $(a)_n = 0$ for all $n > j$ and thus ${}_1F_1$ will terminate at the j^{th} term in (17). In this case ${}_1F_1$ would be a polynomial of degree j and thus it would posses j zeros.

3 Connection between a Confined Harmonic Oscillator and a Free One

Here we consider two specific values of the parameter a given in (9), namely $a = -1$ and -2. The value of λ from (9) is

$$\lambda = 4|a| + N + 2\ell. \tag{19}$$

For the case $a = -1$, the vanishing of ${}_1F_1(a, b, z)$ at the radius of the cavity yields one root only, which is $z_\circ = b = \ell + \frac{N}{2}$. For the ground state ($\ell = 0$) we have $\lambda = N + 4$ and $b = N/2$, and thus the energy, $E = \hbar\omega(2 + \frac{N}{2})$, which is the energy of the state $(n, \ell) = (2, 2)$ for the free harmonic oscillator. The radius of the cavity is given by $\alpha^2 S^2 = z_\circ \Rightarrow S = \sqrt{z_\circ}/\alpha$. It is instructive to relate this radius to a physical property of the free harmonic oscillator. Considering the coordinate $r_\circ$ to be the value of r at which the radial distribution function of the free harmonic oscillator is a maximum. This occurs when the angular momentum has its largest value ($\ell = n$). In that case $n_r = 0$ and thus ${}_1F_1$ is unity, and therefore the radial distribution function, with the help of (12) is

$$D(r) = r^{N-1}|R(r)|^2 \sim r^{N-1+2n} e^{-\alpha^2 r^2},$$

and hence $D(r)$ exhibits a maximum at the value $r_\circ$ obtained by requiring

$$\frac{d}{dr} D(r)\bigg|_{r=r_\circ} = 0.$$

This yields

$$r_\circ = \frac{1}{\alpha}\sqrt{n + \frac{N-1}{2}}. \tag{20}$$

Taking $\alpha = 10^{10} m^{-1}$, implies that both $S = \sqrt{Z_\circ}$ and $r_\circ = \sqrt{n + \frac{N-1}{2}}$, are in units of A°. Table 1 shows some results for the case $a = -1$ and $\ell = 0$.

It is noticed that the radius, S of the cavity which is suitable for the given energy increases as the dimension N increases and gets closer to the value of $r_\circ$ for the free harmonic oscillator, and as N, becomes very large the value of S approaches that of $r_\circ$ and in the infinite dimensional space ($N \to \infty$) the two values become the same (i.e. $S = r_\circ$). Another feature for this case ($a = -1$ and $\ell = 0$) is that the radius of the cavity in dimension N is equal to $r_\circ$ for the free harmonic oscillator in dimension $(N - 3)$, which is easily seen by comparing $S = \sqrt{N/2}/\alpha$ with $r_\circ$ for $n = 2$ in (20).

Table 1 The root $Z_\circ$ of ${}_1F_1$, the radius S of the cavity, $r_\circ$ and the ratio $S/r_\circ$ for different values of N

N	$Z_\circ$	S	$r_\circ$	$S/r_\circ$
2	1	1	1.5811	0.6324
3	1.5	1.2247	1.7321	0.7071
4	2	1.4142	1.8708	0.7559
5	2.5	1.5811	2	0.7906
6	3	1.7321	2.1213	0.8165
7	3.5	1.8708	2.2361	0.8366
8	4	2	2.3452	0.8528
9	4.5	2.1213	2.4495	0.8660
10	5	2.2361	2.5495	0.8771
100	50	7.0711	7.1764	0.9853
1000	500	22.3607	22.3942	0.9985

Table 2 The root $Z_\circ$ of ${}_1F_1$, the radius S of the cavity, $r_\circ$ and the ratio $S/r_\circ$ for different values of N

N	$Z_\circ$	S	$r_\circ$	$S/r_\circ$
2	2	1.4142	1.8708	0.7559
3	2.5	1.5811	2	0.7906
4	3	1.7321	2.1213	0.8165
5	3.5	1.8708	2.2361	0.8366
6	4	2	2.3452	0.8528
7	4.5	2.1213	2.4495	0.8660
8	5	2.2361	2.5495	0.8771
9	5.5	2.3452	2.6458	0.8864
10	6	2.4495	2.7386	0.8944
100	51	7.1414	7.2457	0.9856
1000	501	22.3830	22.4165	0.9985

For $\ell = 1$, we have $\lambda = N + 6$ and $b = 1 + N/2$, and thus the energy $E = \hbar\omega(3 + \frac{N}{2})$ is the same for that of the state $(n, \ell) = (3, 3)$ for the free harmonic oscillator. Table 2 shows some results for the case $a = -1$ and $\ell = 1$.

Again both S and $r_\circ$ increase and get close to each other as the dimension N, increases, and in the limit $N \to \infty$ the value of $S \to r_\circ$. Furthermore, the radius of the cavity in dimension N is equal to $r_\circ$ for the free harmonic oscillator in dimension $(N - 3)$, which can be seen by comparing $S = \sqrt{1 + N/2}/\alpha$ with $r_\circ$ given in (20) for $n = 3$. In addition, comparison between the results in Tables 1 and 2 shows that the confined harmonic oscillator for $\ell = 1$ in dimension N is the same as that for $\ell = 0$ in dimension $(N + 2)$, which can be seen by letting $N \to N + 2$ in S and $r_\circ$ for the $\ell = 0$ case to get exactly the corresponding quantities for the $\ell = 1$ case.

For $a = -2$, the confluent hypergeometric function, ${}_1F_1(a, b, z)$ has two roots. These roots are given by, see (17) and (18), $Z_1 = (b + 1) - \sqrt{b + 1}$ and $Z_2 = (b + 1) + \sqrt{b + 1}$, and thus (19) gives $\lambda = N + 8 + 2\ell$. For the ground state $(\ell = 0)$, we have $\lambda = N + 8$ and $b = N/2$, and thus $E = \hbar\omega(4 + \frac{N}{2})$. The two roots z_1 and z_2 yield two radii for the cavity, S_1 and S_2, which correspond to the states $(n, \ell) = (4, 2)$ and $(4, 4)$. Only the $(4, 4)$ state corresponds to the free harmonic oscillator, as seen from (15) and noting that $\ell = n$, for maximum value of ℓ.

Table 3 Typical results for the roots z_1 and z_2 and their corresponding radii S_1 and S_2. Last column gives $r_\circ$ for the free harmonic oscillator

N	Z_1	S_1	Z_2	S_2	$r_\circ$
2	0.5858	0.7653	3.4142	1.8477	2.1213
3	0.9188	0.9585	4.0811	2.0200	2.2361
4	1.2679	1.1260	4.7321	2.1753	2.3452
5	1.6292	1.2764	5.3708	2.3175	2.4495
6	2	1.4142	6	2.4495	2.5495
7	2.3787	1.5423	6.6213	2.5732	2.6457
8	2.7640	1.6625	7.2361	2.6900	2.7386
9	3.1548	1.7762	7.8452	2.8010	2.8284
10	3.5505	1.8843	8.4495	2.9068	2.9155
100	43.8586	6.6226	58.1414	7.6251	7.3144
1000	478.6170	21.8773	523.3830	22.8776	22.4388

Table 3 shows some typical results for the roots z_1 and z_2, and their corresponding radii S_1 and S_2 of the cavity, using $S_i = \sqrt{Z_i}/\alpha$ with $i = 1, 2$. Results for $r_\circ$, given by (20), have been calculated for comparison purposes. As before, we choose $\alpha = 10^{10} m^{-1}$ so that S_1, S_2 and $r_\circ$ are in units of A°.

The results show that the radius of the cavity becomes closer to $r_\circ$ as the dimension N increases. It is instructive to discuss a general connection between the ground state of a confined harmonic oscillator and some states of the free harmonic oscillator in higher dimensions.

Let $\lambda = N + c$, with c is positive, then $E = \hbar\omega(\frac{c}{2} + \frac{N}{2})$. We consider the following cases:

3.1. $c =$ even integer implies $\frac{c}{2}(\equiv n)$ even or odd. The $n =$ even case yields $\lambda = N + 4 \bmod 4$, $a =$ negative integer, and $E = \hbar\omega(n + \frac{N}{2})$. Owing to (15), this energy corresponds to the states (n, ℓ) for the free harmonic oscillator with $\ell = 2, 4, \ldots, n$ whose number is $n/2$.

The $n =$ odd case yields $\lambda = N + 2 \bmod 4$, $a =$ negative half – integer, and the energy $E = \hbar\omega(n + N/2)$ corresponds to the states (n, ℓ) for the free harmonic oscillator with $\ell = 1, 3, \ldots, n$. whose number is $(n + 1)/2$. Note that the number of these states is equal to number of zeros of ${}_1F_1(a, b, z)$ which is determined by the magnitude of a.

3.2. $c =$ odd integer: using $n = \frac{c}{2}$, the energy $E = \hbar\omega(n + \frac{N}{2})$ with $n = \frac{1}{2}$ odd integer. This energy could be written in two ways:

The first: $E = \hbar\omega(\frac{2n+1}{2} + \frac{N-1}{2})$ which is the same as that of the states $(\frac{2n+1}{2}, \ell)$ of the free harmonic oscillator in $(N - 1)$ dimensions. The values of ℓ are given by

$$\ell = \begin{cases} \dfrac{2n+1}{2}, \dfrac{2n-3}{2}, \ldots, 1, & \text{for } c = 1 \bmod 4, \\ \dfrac{2n+1}{2}, \dfrac{2n-3}{2}, \ldots, 2, & \text{for } c = 3 \bmod 4. \end{cases}$$

The second: $E = \hbar\omega(\frac{2n-1}{2} + \frac{N+1}{2})$ which is the same as that of the states $(\frac{2n-1}{2}, \ell)$ of the free harmonic oscillator in $(N + 1)$ dimensions. The values of ℓ are given by

Table 4 Results for the $(n^{\pm}, \ell)$ states for the free harmonic oscillator in $(N-1)$ and $(N+1)$ dimensions that correspond to the ground state energy of the confined harmonic oscillator in N dimensions

c	(n^{+}, ℓ) in $(N-1)$ dim.	(n^{-}, ℓ) in $(N+1)$ dim.
1	$(1, 1)$	$(0, 0)$
3	$(2, 2)$	$(1, 1)$
5	$(3, 3)$	$(2, 2)$
7	$(4, 4)$	$(3, 3)$
9	$(5, 5)$	$(4, 4)$
11	$(6, 6)$	$(5, 5)$
13	$(7, 7)$	$(6, 6)$
15	$(8, 8)$	$(7, 7)$

$$\ell = \begin{cases} \dfrac{2n-1}{2}, \dfrac{2n-5}{2}, \ldots, 0, & \text{for } c = 1 \bmod 4, \\ \dfrac{2n-1}{2}, \dfrac{2n-5}{2}, \ldots, 1, & \text{for } c = 3 \bmod 4. \end{cases}$$

It is important to note that number of ℓ values in each case equals to $(\frac{2n+3}{4})(= \frac{c+3}{4})$ when $c = 1 \bmod 4$ and $(\frac{2n+1}{4})(= \frac{c+1}{4})$ when $c = 3 \bmod 4$. This is exactly the number of roots of ${}_1F_1(a, b, z)$. This can be explained as follows:

The substitution of $\ell = 0$ and $\lambda = N + c$ in (9) gives that $a = -\frac{c}{4}$. Searching for the roots of ${}_1F_1(a, b, z)$, for a given a and b, using mathematica, shows that for $-j \leq a < -j + 1$ with j integer, the number of roots is just j. This value of j is just the number of ℓ values. To illustrate this connection between the energy of the ground state of the confined harmonic oscillator in N dimensions and the states for the free harmonic oscillator in $(N-1)$, or in $(N+1)$ dimensions, we show some results for several values of c. For simplicity, we defined $n^{\pm} = (\frac{2n \pm 1}{2})$, for the second and third columns of Table 4. We must note here that the roots for ${}_1F_1(a, b, z)$ for a given a and b can be found by using mathematica by searching for z at which ${}_1F_1(a, b, z)$ vanishes, and states $(n, \ell) = (n, n)$ only are included in Table 4.

For computational purposes, we choose $c = 1$ and consider the ground-state for a confined harmonic oscillator in N dimensions. The aim is to find the radius of the cavity for different values of N and its relation to the value of $r_\circ$ at which the radial distribution function for the free harmonic oscillator is maximum. In this case $\lambda = N + 1$ and $a = (N - \lambda)/4 = -0.25$. The ground state energy for the confined oscillator is thus $E = \hbar\omega(\frac{1}{2} + \frac{N}{2})$, and it is just the energy of the state $(0, 0)$ or $(1, 1)$ for the free harmonic oscillator in $(N + 1)$ or $(N - 1)$ dimensions respectively, namely $\hbar\omega(0 + \frac{N+1}{2})$ or $\hbar\omega(1 + \frac{N-1}{2})$. It must be noticed, however, that these two states have the same value for $r_\circ$ as is easily checked by (20), with the result $r_\circ = \sqrt{N/2}/\alpha$. As before, the radius, S of the cavity is given by $\sqrt{Z_\circ}/\alpha$. Results in Table 5 assume $\alpha = 10^{10} m^{-1}$ and thus S and $r_\circ$ are in A°.

3.3. $c = 1/2$ odd integer: For example if $c = 1/2$, then the energy of the confined oscillator becomes $E = \hbar\omega(\frac{1}{4} + \frac{N}{2})$. This energy could be written as $E = \hbar\omega(0 + \frac{N+1/2}{2})$ which is just the ground-state energy of the state $(n', \ell) = (0, 0)$ for the free harmonic oscillator in the fractional dimension $(N + 1/2)$. In general, if we let $c = \frac{n}{2}$ with n being odd, then the energy of the confined harmonic oscillator becomes $E = \hbar\omega(\frac{n}{4} + \frac{N}{2})$. This energy could be written as $E = \hbar\omega(\frac{n-1}{4} + \frac{N+1/2}{2})$ which is exactly the energy of the state $(n', \ell) = (\frac{n-1}{4}, \ell)$ for the free harmonic oscillator in the fractional dimension $(N + 1/2)$. We must mention here that

Table 5 Values for the roots of ${}_1F_1(a,b,z)$ computed numerically and the calculated values of the radius of the cavity and $r_\circ$ for the corresponding states of the free oscillator in different dimensions

N	$Z_\circ$	S	$r_\circ$
2	2.3013	1.5170	1
3	3.1361	1.7709	1.2247
4	3.9129	1.9781	1.4142
5	4.6536	2.1572	1.5811
6	5.3695	2.3172	1.7321
7	6.0667	2.4631	1.8708
8	6.7494	2.5980	2
9	7.4203	2.7240	2.1213
10	8.0814	2.8428	2.2361
20	14.3706	3.7909	3.1623
50	31.9014	5.6481	5
100	59.7394	7.7291	7.0711

n must be $(1+4 \bmod 4)$ in order that the corresponding free harmonic oscillator state, n' to be integer. The connection between states of the confined oscillator in N dimensions with states for the free harmonic oscillator in the fractional dimension $(N+1/2)$ is interesting. The fractional dimensions have been regularly discussed in the literature. The fractional dimension may be viewed as an effective dimension of compactified higher dimensions or a manifestation of non-trivial microscopic lattice structure of space [21].

Applications to fractional dimensions, for example, appear in the study of confinement in low-dimensional systems [11]. Other examples are the modeling of exciton–phonon interactions [25]. Recently, algebraic approach to quantum mechanics in fractional dimensions has been reported [15].

4 Energy Levels and Eigenfunctions for a Confined Harmonic Oscillator

The purpose of this section is to find numerically the energy levels for selected eigenstates for the confined N-dimensional harmonic oscillator. Given the radius S of the cavity, we search for the values of the parameter a, at which the confluent hypergeometric function ${}_1F_1(a,b,z)$ vanishes, for given value of ℓ. Letting $a_{n\ell}$ the n^{th} value of a, for a given ℓ, at which ${}_1F_1$ has its n^{th} zero at $Z_\circ = \alpha^2 S^2$, (19) then yields $\lambda_{n\ell} = 4|a_{n\ell}| + N + 2\ell$ and thus the energy eigenvalues are then given by

$$E_{n\ell} = \hbar\omega\frac{\lambda}{2} = 2|a_{n\ell}| + \ell + \frac{N}{2}. \tag{21}$$

The eigenfunctions are therefore given by (see (12))

$$\psi_{n\ell m}(r,\{\theta_i\}) = A_{n\ell} r^\ell e^{-z_\circ r^2/2S^2}\, {}_1F_1\left[\frac{1}{2}\left(\ell+\frac{N}{2}\right) - \frac{\lambda_{n\ell}}{4}, \ell+\frac{N}{2}, Z_\circ r^2/S^2\right]. \tag{22}$$

The eigenvalues $E_{n\ell}$ to be computed numerically, using mathematica, are E_{10}, E_{21}, and E_{32}. This means we search for the first zero of ${}_1F_1$ by varying a at a given S and dimension N for $\ell = 0$. The value of a at which ${}_1F_1$ vanishes is just a_{10}, and hence (21) gives E_{10}. Similarly, in order to find E_{21} and E_{32} we search for a_{21} and a_{32} that correspond to the second and third zeros for $\ell = 1$ and $\ell = 2$ respectively. For comparison purposes, this has been done

Table 6 Energy eigenvalues for the energy levels $E_{n\ell}$ obtained numerically, using mathematica, for N-dimensional harmonic oscillators confined in two cavities of radii S_1 and S_2

N	$E_{n\ell}$ for $S_1 = 2$ Å			$E_{n\ell}$ for $S_2 = \sqrt{10}$ Å		
	E_{10}	E_{21}	E_{32}	E_{10}	E_{21}	E_{32}
2	1.1222	6.8258	17.5774	1.0008	4.1622	8.5462
3	1.7648	8.2795	19.6974	1.5028	4.7584	9.4180
4	2.4718	9.5828	21.9063	2.0070	5.3836	10.3300
5	3.2469	11.0934	24.2028	2.5156	6.0404	11.2782
6	4.0926	12.6894	26.5862	3.0312	6.7284	12.2610
7	5.0101	14.3700	29.0550	3.5514	7.4484	13.2786
8	6.0000	16.1336	31.6092	4.0886	8.2000	14.3304
9	7.0629	17.9792	34.2476	4.6368	8.9832	15.4156
10	8.1987	19.9063	36.9696	5.2008	9.7976	16.5342

for two different values of S, namely, $S_1 = 2$ Å and $S_2 = \sqrt{10}$ Å. These results are reported in Table 6.

The numerical method used to obtain the results in Table 6 can be used for the energy of other energy levels for any size of the cavity. In addition, these results have the advantage that they are considered as a reference to test the accuracy or the validity of approximate methods that may be applied to such problems. The results also show that the energy eigenvalue $E_{n\ell}$ for the cavity of radius S_1 $(= 2$ Å) is larger than its corresponding value for the cavity of radius S_2 $(= \sqrt{10}$ Å). This may be due to the fact that a larger work must be needed to compress a free harmonic oscillator into a smaller size. It should be clear that the results for E_{10} for the case of cavity radius S_2 closely correspond to the state $(0, 0)$ of the free harmonic oscillator, namely $E_\circ^N = \hbar\omega(\frac{N}{2})$. To examine this point in more detail and its correlation to the size of the cavity, we consider a cavity of radius $S = 4$ Å. Numerical search is performed as before in order to compute the energy eigenvalues E_{10} and E_{20}. This means we search for a_{10} at which ${}_1F_1(a, \frac{N}{2}, 16)$ has its first zero, and then E_{10} is computed from (21) with $\ell = 0$. Similarly, we compute E_{20}, but the search is performed for the second zero of ${}_1F_1$. We report these results in Table 7.

It is observed that E_{10} corresponds to the state $(n, \ell) = (0, 0)$ for the free N-dimensional harmonic oscillator with energy $E_\circ^N = \hbar\omega(\frac{N}{2})$, and E_{20} corresponds to the state (2, 1) with energy $E_2^N = \hbar\omega(2 + N/2)$.

Table 7 also includes the percentage of the difference between E_{10} and $E_\circ^N$ as well as between E_{20} and E_2^N. These results show that the percentages are really small for low dimension, but they start to increase as N increases. This is due to the increase of $r_\circ$ at which the radial distribution function is maximum, and thus the effect of the boundaries becomes more profound. One also observes that the effect of the boundary is relatively larger on the energy level E_{20} compared to its effect on E_{10}. This is so since the former corresponds to the state $(2, 2)$ while the latter corresponds to the state $(0, 0)$. This implies that the value of $r_\circ$ for the state $(2, 2)$ is larger than its value for the state $(0, 0)$ as is clearly noticed form (20), and thus the effect of the boundary is expected to be larger on the state $(2, 2)$.

Table 7 Computed results for the energy eigenvalues E_{10} and E_{20} for a confined N-dimensional harmonic oscillator in a cavity of radius $S = 4$ Å, and their percentages compared to $E_\circ^N$ and E_2^N of the free oscillator

N	E_{10}	E_{20}	$\frac{E_{10}-E_\circ^N}{E_2^N} \times 100\%$	$\frac{E_{20}-E_2^N}{E_2^N} \times 100\%$
2	1.000003	3.00064	0.00015	0.02133
3	1.500015	3.50170	0.00100	0.04857
4	2.00005	4.00379	0.00250	0.09485
5	2.50014	4.50832	0.00576	0.18488
6	3.00037	5.01605	0.01220	0.32100
7	3.50084	5.52870	0.02400	0.52180
8	4.00178	6.04800	0.04450	0.80000
9	4.50349	6.57600	0.07760	1.16923
10	5.00644	7.11400	0.12880	1.62857
20	10.28000	13.42000	2.8000	11.000
50	33.66800	42.29000	34.672	56.630

5 Mapping of Energy Eigenvalues

It is interesting to compare energy eigenvalues in dimension N with those in different dimensions. For example, the energies for $\ell = 0$ in dimension $N = 5$ correspond to the solutions for the roots of ${}_1F_1(\frac{5}{4} - \frac{\lambda}{4}, \frac{5}{4}, \alpha^2 r^2)$, i.e. they are identical to the three-dimensional solutions for the case $\ell = 1$, see (12). Similarly for $N = 5$ and $\ell = 1$, one has all the three-dimensional solutions for $\ell = 2$. However, the $N = 3$ and $\ell = 0$ solutions are missing for the $N = 5$ case. Similarly for $N = 7$, the $\ell = 0$ and $\ell = 1$, $N = 3$ levels are missing but the $\ell = 2$ levels correspond exactly to the $N = 7$, $\ell = 0$ levels. A similar analysis can be carried out for $N = 2$, 4, 6, etc. To put this correspondence in more general terms, we consider the two different parities of N separately:

For $N =$ odd and angular momentum ℓ, the energies correspond to solutions for the roots of ${}_1F_1(\frac{1}{2}(\ell + \frac{N}{2}) - \frac{\lambda}{4}, \ell + \frac{N}{2}, \alpha^2 r^2)$.

These correspond exactly to those either for dimension $(N - 2)$ and angular momentum $\ell' = \ell + 1$, or they correspond to the three dimensional case with $\ell' = \ell + (N - 3)/2$. For $N =$ even and angular momentum ℓ, the energies correspond to those in dimension $(N - 2)$ with angular momentum $\ell' = \ell + 1$, or they correspond to those for two-dimensional case $(N = 2)$ with $\ell' = \ell + (N - 2)/2$.

6 Conclusion

In this paper, we have investigated the confined harmonic oscillator in the N-dimensional space. The energy eigenvalues were computed by searching for the roots of the hypergeometric function ${}_1F_1(a, b, z)$ at which the wave function vanishes at the surface of the confining cavity.

We first examined the special cases $a = -1$ with $\ell = 0$ and 1 and $a = -2$ with $\ell = 0$, and found the roots of ${}_1F_1$ analytically. In these two cases, we discussed the correspondence of the energy eigenvalues to states $(n, \ell) = (n, n)$ for the free harmonic oscillator, and pointed out that the radius of the cavity increases and gets closer to $r_\circ$, given in (20), as the dimension

N increases. We then investigated the general case $\lambda = N + c$ in some details. For example, by letting $c = \frac{n}{2}$, our results showed that the energy eigenvalues correspond to the states (n, ℓ) for the free harmonic oscillator with $\ell = 2, 4, \ldots, n$ when n is even and $\ell = 1, 3, \ldots, n$ when $n =$ odd. We have observed that the number of ℓ values is just the number of roots of ${}_1F_1(a, b, z)$. The other case we have considered is when $c =$ odd integer which implies that $n =$ half-odd integer. In this case, we have found that the energy eigenvalues correspond to those for the free harmonic oscillator states $(\frac{2n+1}{2}, \ell)$ in $(N-1)$ dimension or to the state $(\frac{2n-1}{2}, \ell)$ in $(N+1)$ dimension. We have also reported numerical results for the radius of the cavity when $c = 1$. Moreover, we have examined the case $c =$ half-odd integer, i.e. $c = n/2$ with $n =$ odd. Here, it was demonstrated that the energy eigenvalues for a confined oscillator in dimension N are identical to those for the free oscillator in the fractional dimension $(N + 1/2)$.

We have used mathematica to find the energy eigenvalues for a confined harmonic oscillator in a given radius of the cavity. This was done by searching for the $a_{n\ell}$ root of ${}_1F_1$ which means finding the n^{th} root of ${}_1F_1$ for a given ℓ, which is then used to determine the energy $E_{n\ell}$. In particular, we computed E_{10}, E_{21}, and E_{32} for two different cavities and found that these values are greater in the smaller cavity. Furthermore, we computed E_{10} and E_{20} for a cavity of radius $S = 4$ Å and found that their values are very close to those for the states $(0, 0)$ and $(2, 2)$ of the free harmonic oscillator respectively. Finally, we examined the mapping between energy eigenvalues for the confined oscillator in different dimensions. We have reported that the energies for dimension N and angular momentum ℓ are exactly the same to those for a cavity in dimension $(N-2)$ and angular momentum $(\ell+1)$ or, alternatively, they are identical to those of a cavity in three dimensions with angular momentum $(\ell + \frac{N-3}{2})$ when N is odd or identical to those of a cavity in two dimensions with angular momentum $(\ell + \frac{N-2}{2})$.

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