An approximate κ state solutions of the Dirac equation for the generalized Morse potential under spin and pseudospin symmetry

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By using an improved approximation scheme to deal with the centrifugal (pseudo-centrifugal) term, we solve the Dirac equation for the generalized Morse potential with arbitrary spin-orbit quantum number κ. In the presence of spin and pseudospin symmetry, the analytic bound state energy eigenvalues and the associated upper- and lower-spinor components of two Dirac particles are found by using the basic concepts of the Nikiforov-Uvarov method. We study the special cases when $\kappa = \pm 1$ ($l = \tilde{l} = 0$, s-wave), the non-relativistic limit and the limit when α becomes zero (Kratzer potential model). The present solutions are compared with those obtained by other methods. © *2011 American Institute of Physics*. [doi:10.1063/1.3583553]

I. INTRODUCTION

The simplest modified Morse potential model suggested by Deng and Fan[1] and related to the Manning-Rosen potential[2] (also called Eckart potential by some authors[3]) is anharmonic potential defined by

$$V(r) = D \left(1 - \frac{b}{e^{\alpha r} - 1} \right)^2 , \, b = e^{\alpha r_e} - 1, \tag{1}$$

where $r \in (0, \infty)$, and the three positive parameters D, r_e, and α denote the dissociation energy, the equilibrium inter-nuclear distance, and the range of the potential well, respectively. The above potential is used to describe diatomic molecular energy spectra and electromagnetic transitions and is the true internuclear potential in diatomic molecules with the same behaviour for $r \to 0$.[4] The above potential was called a generalized Morse potential (GMP) model and illustrated in Fig. 1 for various values of potential parameters. As stated in Ref. 4, the Morse potential and the GMP are very close to each other for large values of r_e ($\alpha = 1$) in the regions $r \sim r_e$ and $r > r_e$ but are very different at $r \sim 0$. Further, if the two potentials are deep ($D \gg 1$), they could be well approximated by a harmonic oscillator in the region $r \sim r_e$ (see Fig. 1 in Ref. 4). To describe the vibrational spectra of diatomic molecules the potential curve $V(r)$ is approximated by a sum of three Morse functions.[5] The approach has the advantage of being more flexible than the simple Morse potential while preserving the correct asymptotical $V(\infty) = const$.[5] The potential model (1) is a special case of the five-parameter exponential-type potential model.[6,7] The exact solvability of the s-wave ($l = 0$ case) bound state energy eigenvalues and eigenfunctions of the GMP is due to the fact that it belongs to the class of the Eckart potential, a member of the hypergeometric Natanzon potentials which can be solved algebraically by means of $SO(2, 2)$ symmetry algebra[4] and $SO(2, 1)$ algebra.[8] The X-H stretching motion in small molecules has been treated by the potential model (1).[9] Moreover, the approximated l-wave bound state solutions[10] of the Schrödinger equation has been solved using the conventional approximation scheme suggested by Greene and Aldrich[11] to deal with the centrifugal barrier term $l(l + 1)/r^2$, singularity at $r = 0$. The exact analytic expressions for matrix elements

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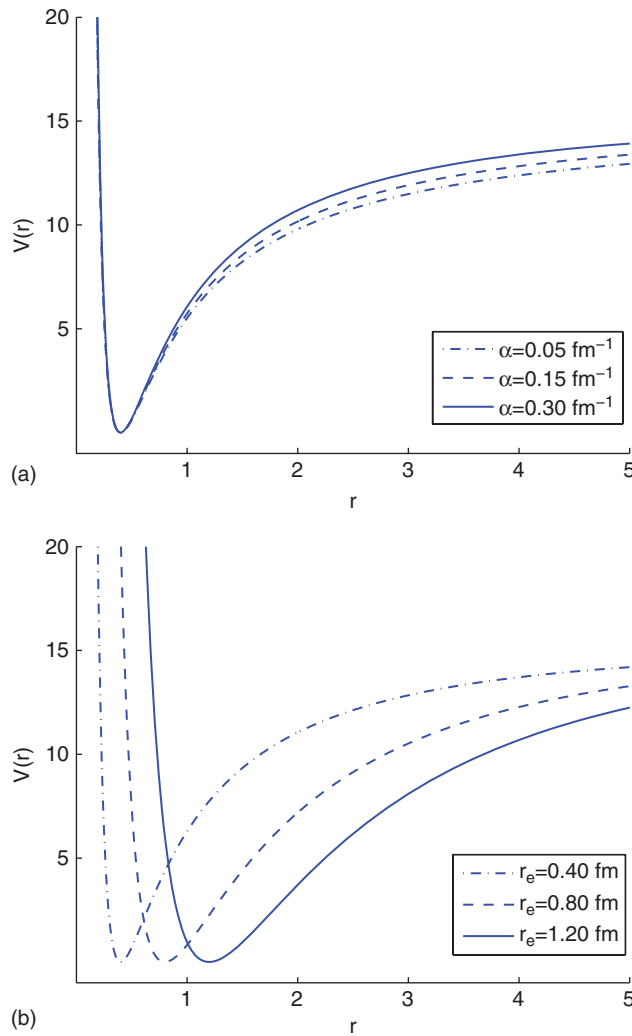a)Electronic mail: sikhdair@neu.edu.tr.

FIG. 1. (Color online) The GMP model with $D = 15$ fm^{-1} for (a) various potential ranges $\alpha = 0.05, 0.15, 0.30$ fm^{-1} along with $r_e = 0.40$ fm, and (b) various equilibrium inter-nuclear distances $r_e = 0.40, 0.80, 1.20$ fm along with $\alpha = 0.40$ fm^{-1}.

of positive integral powers of the coordinate[12] and the quasi-one-dimensional system of DNA (Ref. [13]) have also been studied.

    To study the relativistic effects and corrections of the molecular Morse potential,[4] the Dirac equation has been solved for attractive scalar $S(r)$ and repulsive vector $V(r)$ Morse potentials under pseudospin symmetry in the nuclear theory using the Pekeris approximation.[14] Recently, the approximate bound state solutions of the pseudospin and spin symmetric Dirac equation with the GMP has been calculated using an improved approximation scheme to deal with the centrifugal (pseudo-centrifugal) term and by employing the basic concept of the supersymmetric shape invariance formalism (cf. Ref. [15] and the references therein).

    Many authors have investigated approximately the solution of the Dirac equation with a few potential models such as the Hulthén potential,[16] the Hulthen potential including Coulomb-like tensor potential,[17] the generalized Woods-Saxon potential,[18,19] the Eckart potential,[20] the Morse potential,[14,21] the Pöschl-Teller potential,[22] the Manning-Rosen potential,[2,23] the hyperbolic potential,[24] the Rosen-Morse potential,[25] the pseudoharmonic potential,[26] and the Kratzer potential connected with an angle-dependent potential,[27] etc. within the framework of various methods.

    For a diatomic molecular model (DMM) instead of the nuclear model (NM), the reduced mass definition can be considered. If the nuclei have masses m_1 and m_2, the reduced mass is defined $\mu = m_1 m_2 / (m_1 + m_2)$ and hence the DMM can be included to the spin symmetry and the pseudospin

symmetry concepts.[28] Ginocchio[29–31] showed that the spin symmetry occurs when the difference potential between the vector potential $V(r)$ and scalar potential $S(r)$ is a constant (i.e., $\Delta(r) = V(r) - S(r) = C_s = $ constant) and the pseudospin symmetry occurs when the sum potential of the vector potential $V(r)$ and scalar potential $S(r)$ is a constant (i.e., $\Sigma(r) = V(r) - S(r) = C_{ps} = $ constant). The spin symmetry concept[32] is particularly relevant for mesons.[33] The pseudospin symmetry concept[34,35] in nuclear physics refers to the quasi-degeneracy of single-nucleon doublets and can be characterized with the non-relativistic quantum numbers $(n, l, j = l + 1/2)$ and $(n - 1, l + 2, j = l + 3/2)$, where n, l, and j are the single-nucleon radial, orbital, and total angular momentum quantum numbers, respectively. Alhaidari et al.[36] investigated in detail the physical interpretation on the three-dimensional Dirac equation in the cases of spin symmetry limitation $\Delta(r) = 0$ and pseudospin symmetry limitation $\Sigma(r) = 0$. In real nuclei, $\Sigma(r) \neq$ constant and pseudospin symmetry is only an approximation. The quality of the pseudospin symmetry approximation depends on the competition between the pseudo-centrifugal potential and the pseudospin orbital potential.[37]

Recently, in the framework of the spin symmetry $S(r) \sim V(r)$ and pseudospin symmetry $S(r) \sim -V(r)$, the bound state energy eigenvalues and associated upper- and lower-spinor wave functions are investigated by means of the Nikiforov-Uvarov (NU) method.[38] We have approximately solved the Dirac equation for the Rosen-Morse potential[25] with spin and pseudospin symmetry for any κ state and found the eigenvalue equation and corresponding two-component spinors within the framework of an approximation to the term proportional to $1/r^2$. We have also solved the $(3 + 1)$ dimensional Dirac equation for a particle trapped in the spherically symmetric generalized WS potential under the conditions of exact spin and pseudospin symmetry combined with approximation for the spin-orbit centrifugal (pseudo-centrifugal) term, and calculated the two-component spinor wave functions and the energy eigenvalues for any arbitrary spin-orbit κ bound states.[18]

In principle, the solution of the Schrödinger equation for the GMP model can be used to describe the motion of the nucleons in the mean field produced by the interactions between nuclei. However, the Dirac equation successfully merges quantum mechanics with special relativity and is considered to be the natural transition to quantum field theory. It provides a natural description of the electron spin, predicts the existence of antimatter, and is able to reproduce accurately the spectrum of the hydrogen atom. It also predicts some peculiar phenomena such as Klein's paradox and unexpected quivering motion of free relativistic quantum particle which are key examples for understanding relativistic quantum effects, but are difficult to observe in real particles. In order to to understand the origin of spin and pseudospin symmetry, we need to take into consideration the motion of the nucleons in a relativistic mean field and consider the Dirac equation.[31] We attempt to study the approximate solution of the Dirac equation for the GMP with non-zero spin-orbit quantum numbers κ by employing an improved approximation scheme (see, e.g., Refs. 17, 39, and 40] and the references therein) to deal with the spin-orbit centrifugal, $\kappa(\kappa + 1)/r^2$ (pseudo-centrifugal, $\kappa(\kappa - 1)/r^2$) barrier term. The inter-relation between the GMP with the Oscillator and Morse potentials investigated in Ref. 4 and the recent relativistic treatments of the Morse and Oscillator potentials in Refs. 14 and 26 are some motivations for the present study. The simple transformation of the Dirac equation into the Schrödinger-like equation and the success in studying the approximated bound state solutions of the Dirac equation with various potential models [14,17,18,25,26] also make the solution possible.

We tend to show that the new scheme of parametric generalization of the NU method [41] given in Appendix A is a powerful tool for solving a second order differential equation by turning it into a hypergeometric type equation.[38] The advantage of employing the NU method, in the present work, is that it can be used to find the bound state energy spectra and the corresponding spinor wave functions under the condition of spin symmetry and pseudospin symmetry concept for any κ state in a very simple way.

This paper is organized as follows. In Sec. II, we investigate the bound state energy equation and the corresponding two-component spinor wave functions under the condition of spin symmetry and pseudospin symmetry concept for the GMP model by employing a parametric generalization of the NU method. In Sec. III, we study some special cases like the $s(\tilde{s})$-wave cases $\kappa = \pm 1$ ($l = \tilde{l} = 0$), the non-relativistic limit and the $\alpha \to 0$ (Kratzer potential). In Sec. IV, we present some numerical results to the non-relativistic and relativistic numerical energy levels for GMP and Kratzer models. The relevant conclusions are given in Sec. V.

II. DIRAC BOUND STATE SOLUTIONS

The Dirac equation for fermionic massive spin-1/2 particles moving in an attractive scalar potential $S(r)$ and a repulsive vector potential $V(r)$ is given by[42]

$$\left[c\alpha \cdot \mathbf{p} + \beta \left(Mc^2 + S(r)\right) + V(r) - E\right] \psi_{n\kappa}(\mathbf{r}) = 0, \ \psi_{n\kappa}(\mathbf{r}) = \psi(r, \theta, \phi), \tag{2}$$

where E is the relativistic energy of the system, m is the mass of a particle, $\mathbf{p} = -i\hbar\nabla$ is the momentum operator, and α and β are 4×4 Dirac matrices, i.e.,

$$\alpha = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \ \beta = \begin{pmatrix} \mathbf{I} & 0 \\ 0 & -\mathbf{I} \end{pmatrix}, \ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{3}$$

where $\mathbf{I}$ denotes the 2×2 identity matrix and σ_i are the three-vector Pauli spin matrices. For a spherical symmetrical nuclei, the total angular momentum operator of the nuclei $\mathbf{J}$ and spin-orbit matrix operator $\mathbf{K} = -\beta\left(\sigma \cdot \mathbf{L} + \mathbf{I}\right)$ commute with the Dirac Hamiltonian, where $\mathbf{L}$ is the orbital angular momentum operator. The spinor wave functions can be classified according to the radial quantum number n and the spin-orbit quantum number κ and can be written using the Pauli-Dirac representation in the following forms:

$$\psi_{n\kappa}(\mathbf{r}) = \begin{pmatrix} f_{n\kappa}(\mathbf{r}) \\ g_{n\kappa}(\mathbf{r}) \end{pmatrix} = \frac{1}{r} \begin{pmatrix} F_{n\kappa}(r) Y_{jm\kappa}^l(\widehat{r}) \\ i G_{n\kappa}(r) Y_{jm(-\kappa)}^{\widetilde{l}}(\widehat{r}) \end{pmatrix}, \tag{4}$$

where the upper- and lower-spinor components $F_{n\kappa}(r)$ and $G_{n\kappa}(r)$ are the real square-integral radial wave functions, $Y_{jm\kappa}^l(\widehat{r})$ and $Y_{jm(-\kappa)}^{\widetilde{l}}(\widehat{r})$ are the spin spherical harmonic functions coupled to the total angular momentum j and its projection m on the z axis and $\kappa(\kappa+1) = l(l+1)$ and $\kappa(\kappa-1) = \widetilde{l}(\widetilde{l}+1)$. The quantum number κ is related to the quantum numbers for spin symmetry l and pseudospin symmetry $\widetilde{l}$ as

$$\kappa = \begin{cases} -(l+1) = -\left(j+\frac{1}{2}\right), & (s_{1/2}, p_{3/2}, etc.), \ j = l + \frac{1}{2}, & \text{aligned spin } (\kappa < 0), \\ +l = +\left(j+\frac{1}{2}\right), & (p_{1/2}, d_{3/2}, etc.), \ j = l - \frac{1}{2}, & \text{unaligned spin } (\kappa > 0), \end{cases} \tag{5}$$

and the quasi-degenerate doublet structure can be expressed in terms of a pseudospin angular momentum $\widetilde{s} = 1/2$ and pseudo-orbital angular momentum $\widetilde{l}$ which is defined as

$$\kappa = \begin{cases} -\widetilde{l} = -\left(j+\frac{1}{2}\right), & (s_{1/2}, p_{3/2}, etc.), \ j = \widetilde{l} - 1/2, & \text{aligned spin } (\kappa < 0), \\ +(\widetilde{l}+1) = +\left(j+\frac{1}{2}\right), & (d_{3/2}, f_{5/2}, etc.), \ j = \widetilde{l} + 1/2, & \text{unaligned spin } (\kappa > 0), \end{cases} \tag{6}$$

where $\kappa = \pm 1, \pm 2, \ldots$. For example, $(1s_{1/2}, 0d_{3/2})$ and $(2p_{3/2}, 1f_{5/2})$ can be considered as pseudospin doublets.

Upon direct substitution of Eq. (4) into Eq. (7b), we can obtain two radial coupled Dirac equations for the two spinor components as follows:

$$\left(\frac{d}{dr} + \frac{\kappa}{r}\right) F_{n\kappa}(r) = \left[Mc^2 + E_{n\kappa} - \Delta(r)\right] G_{n\kappa}(r), \tag{7a}$$

$$\left(\frac{d}{dr} - \frac{\kappa}{r}\right) G_{n\kappa}(r) = \left[Mc^2 - E_{n\kappa} + \Sigma(r)\right] F_{n\kappa}(r), \tag{7b}$$

where $\Delta(r) = V(r) - S(r)$ and $\Sigma(r) = V(r) + S(r)$ are the difference and sum potentials, respectively.

Under the spin symmetry (i.e., $\Delta(r) = C_s = $ constant), one can eliminate $G_{n\kappa}(r)$ in Eq. (7a)), with the aid of Eq. (7b), to obtain a second-order differential equation for the upper-spinor component as follows:

$$\left[-\frac{d^2}{dr^2} + \frac{\kappa(\kappa+1)}{r^2} + \frac{1}{\hbar^2 c^2}\left(Mc^2 + E_{n\kappa} - C_s\right)\Sigma(r)\right] F_{n\kappa}(r)$$

$$= \frac{1}{\hbar^2 c^2}\left(E_{n\kappa}^2 - M^2 c^4 + C_s\left(Mc^2 - E_{n\kappa}\right)\right) F_{n\kappa}(r), \tag{8}$$

and the lower-spinor component is obtained from Eq. (7a) as

$$G_{n\kappa}(r) = \frac{1}{Mc^2 + E_{n\kappa} - C_s}\left(\frac{d}{dr} + \frac{\kappa}{r}\right)F_{n\kappa}(r), \tag{9}$$

where $E_{n\kappa} \neq -Mc^2$, only real positive energy states exist when $C_s = 0$ (exact spin symmetry). On the other hand, under the pseudospin symmetry (i.e., $\Sigma(r) = C_{ps} = $ constant), one can eliminate $F_{n\kappa}(r)$ in Eq. (7b), with the aid of Eq. (7a), to obtain a second-order differential equation for the lower-spinor component as follows:

$$\left[-\frac{d^2}{dr^2} + \frac{\kappa(\kappa-1)}{r^2} - \frac{1}{\hbar^2 c^2}\left(Mc^2 - E_{n\kappa} + C_{ps}\right)\Delta(r)\right]G_{n\kappa}(r)$$

$$= \frac{1}{\hbar^2 c^2}\left(E_{n\kappa}^2 - M^2 c^4 - C_{ps}\left(Mc^2 + E_{n\kappa}\right)\right)G_{n\kappa}(r), \tag{10}$$

and the upper-spinor component $F_{n\kappa}(r)$ is obtained from Eq. (7b) as

$$F_{n\kappa}(r) = \frac{1}{Mc^2 - E_{n\kappa} + C_{ps}}\left(\frac{d}{dr} - \frac{\kappa}{r}\right)G_{n\kappa}(r), \tag{11}$$

where $E_{n\kappa} \neq Mc^2$, only real negative energy states exist when $C_{ps} = 0$ (exact pseudospin symmetry).[16] It is worthy to note that the reality and finiteness of our solutions demand that the upper and lower radial components should satisfy the essential boundary conditions: $F_{n\kappa}(0) = G_{n\kappa}(0) = 0$ and $F_{n\kappa}(\infty) = G_{n\kappa}(\infty) = 0$.

A. Spin symmetry solutions of the GMP model

At first, we investigate the spin symmetry by taking the $\Sigma(r) = 2V(r) \rightarrow V_{GMP}(r)$ as mentioned in Refs. 17, 36, and 43. This choice enables us to reduce the resulting relativistic solutions into their non-relativistic limit under appropriate transformations. From Eq. (8), we can see that the energy eigenvalues, $E_{n\kappa}$, depend only on n and l, i.e., $E_{n\kappa} = E(n, l(l+1))$. For $l \neq 0$, the states with $j = l \pm 1/2$ are degenerate. This is a $SU(2)$ spin symmetry. Following Refs. 22–25, we impose the GMP (Refs. 1 and 4) as the $\Sigma(r)$, i.e.,

$$\Sigma(r) = D\left(1 - \frac{b}{e^{\alpha r} - 1}\right)^2, \tag{12}$$

leads us to obtain a Schrödinger-like equation in the form:

$$\left[\frac{d^2}{dr^2} - \frac{\kappa(\kappa+1)}{r^2} - \alpha^2 \nu_1^2\left(1 - \frac{be^{-\alpha r}}{1 - e^{-\alpha r}}\right)^2 + \alpha^2 \omega_1^2\right]F_{n\kappa}(r) = 0 \tag{13}$$

$$\nu_1^2 = \frac{1}{\alpha^2 \hbar^2 c^2}\left(Mc^2 + E_{n\kappa} - C_s\right)D, \quad \omega_1^2 = \frac{1}{\alpha^2 \hbar^2 c^2}\left[E_{n\kappa}^2 - M^2 c^4 + C_s\left(Mc^2 - E_{n\kappa}\right)\right], \tag{14}$$

where $\kappa(\kappa+1) = l(l+1)$, $\kappa = l$ for $\kappa < 0$, and $\kappa = -(l+1)$ for $\kappa > 0$. The exact solution of the above equation is possible only for the s-wave case ($\kappa = -1$) due to the centrifugal term $\kappa(\kappa+1)/r^2$. However, if κ is not too large, the case of the vibrations of small amplitude about the minimum, we attempt to use the following improved new approximation scheme to deal with the centrifugal (pseudo-centrifugal) term, near the minimum point $r = r_e$ (i.e., $x = 0$), (cf. Refs. 17, 23, 40, and 44–48):

$$\frac{1}{r^2} \approx \alpha^2\left[d_0 + \frac{e^{-\alpha r}}{(1 - e^{-\alpha r})^2}\right] = \alpha^2\left[d_0 + \frac{1}{(\alpha r)^2} - \frac{1}{12} + \frac{(\alpha r)^2}{240} - \frac{(\alpha r)^4}{6048} + \frac{(\alpha r)^6}{172800} + O\left((\alpha r)^8\right)\right]. \tag{15}$$

When $\alpha r \ll 1$, the value of the dimensionless constant $d_0 = 1/12$ has simply determined by the above series expansion and α takes the unit of reciprocal of r. The present approximation was shown

to be more powerful than the usual approximation.[40] Obviously, the above approximation to the centrifugal (pseudo-centrifugal) term turns to $1/r^2$ when the parameter α goes to zero as

$$\lim_{\alpha \to 0} \left[\alpha^2 \left(d_0 + \frac{1}{e^{\alpha r} - 1} + \frac{1}{(e^{\alpha r} - 1)^2} \right) \right] = \frac{1}{r^2}, \tag{16}$$

which shows that the usual approximation is the limit of our approximation (cf. e.g.,[17,40] and the references therein).

We introduce the following new dimensionless parameter, $z(r) = e^{-\alpha r} \in [0, 1]$, which maintains the finiteness of the transformed wave functions on the boundary conditions. Thus, substituting Eq. (15) into Eq. (13), we obtain the following Schrödinger-like equation satisfying $F_{n\kappa}(r)$,

$$\left(\frac{d^2}{dz^2} + \frac{(1-z)}{z(1-z)} \frac{d}{dz} \right) F_{n\kappa}(z)$$

$$+ \frac{1}{z^2(1-z)^2} \left\{ - \left[(2+b)\, b\nu_1^2 + \varepsilon_{n\kappa}^2 \right] z^2 + \left[2 \left(b\nu_1^2 + \varepsilon_{n\kappa}^2 \right) - \kappa\,(\kappa+1) \right] z - \varepsilon_{n\kappa}^2 \right\} F_{n\kappa}(z) = 0, \tag{17}$$

where

$$\varepsilon_{n\kappa}^2 = \nu_1^2 - \omega_1^2 + \kappa\,(\kappa+1)\, d_0, \tag{18}$$

and $F_{n\kappa}(r) = F_{n\kappa}(z)$. In order to clarify the parametric generalization of the NU method,[25] let us take the following general form of a Schrödinger-like equation written for any potential,

$$\frac{d^2 \psi_n(z)}{dz^2} + \frac{\widetilde{\tau}(z)}{\sigma(z)} \frac{d\psi_n(z)}{dz} + \frac{\widetilde{\sigma}(z)}{\sigma^2(z)} \psi_n(z) = 0, \tag{19}$$

satisfying the wave functions

$$\psi_n(z) = \phi(z) y_n(z), \tag{20}$$

where

$$\widetilde{\tau}(z) = c_1 - c_2 z, \tag{21}$$

and

$$\sigma(z) = z\,(1 - c_3 z) \text{ and } \widetilde{\sigma}(z) = -Az^2 + Bz - C \tag{22}$$

are two polynomials at most of first- and second-degree, respectively. Furthermore, when Eq. (17) is compared with its counterpart Eq. (19), we can obtain the specific values for the constants c_i ($i = 1, 2, 3$) along with A, B, and C. Now, following the NU method[38] and making the substitution of Eqs. (21) and (22), we can obtain general forms for the polynomials $\pi(z)$ and $\tau(z)$, the root of the parameter k, the eigenvalues equation and the wave functions $\phi(z)$ and $y_n(z)$ expressed in terms of the constants c_i ($i = 4, 5, \ldots, 13$) as shown in Appendix A (cf. Refs. 18, 21, 25, and 40). Therefore, the task of computing the energy eigenvalues and the corresponding wave functions of Eq. (17) within the framework of the parametric generalization of the NU method is relatively easy and straightforward. It is explained in the following steps. Firstly, we need to find the specific values for the parametric constants c_i ($i = 4, 5, \ldots, 13$) by means of the relation A1 of Appendix A. The values of all these constants c_i ($i = 1, 2, \ldots, 13$) together with A, B, and C are therefore displayed in Table I for the GMP model. Secondly, using the relations (A2–A5), the analytic forms of the essential polynomials $\pi(z)$ and $\tau(z)$ along with the root k, required by the method,[41] can also be found as

$$\pi(z) = -\frac{z}{2} - \frac{1}{2} \left[\left(\sqrt{(1+2\kappa)^2 + 4b^2\nu_1^2} + 2\varepsilon_{n\kappa} \right) z - 2\varepsilon_{n\kappa} \right], \tag{23}$$

$$k = 2b\nu_1^2 - \kappa\,(\kappa+1) - \varepsilon_{n\kappa} \sqrt{(1+2\kappa)^2 + 4b^2\nu_1^2}, \tag{24}$$

$$\tau(z) = 1 + 2\varepsilon_{n\kappa} - 2 \left(1 + \varepsilon_{n\kappa} + \frac{1}{2}\sqrt{(1+2\kappa)^2 + 4b^2\nu_1^2} \right) z, \tag{25}$$

TABLE I. Specific values of the constants in the solution of the GMP under spin symmetry.

Constant	Constant
$c_1 = 1$	$c_2 = 1$
$c_3 = 1$	$c_4 = 0$
$c_5 = -\frac{1}{2}$	$c_6 = \frac{1}{4}\left[1 + 4\left((2+b)\xi_1 - \widetilde{E}_{n\kappa}\right)\right]$
$c_7 = 2\left(\widetilde{E}_{n\kappa} - \xi_1\right) + \kappa(\kappa+1)$	$c_8 = -\widetilde{E}_{n\kappa}$
$c_9 = \frac{1}{4}\left[(1+2\kappa)^2 + 4b\xi_1\right]$	$c_{10} = 2\eta_1 = 2i\sqrt{\widetilde{E}_{n\kappa}}$
$c_{11} = \sqrt{(1+2\kappa)^2 + 4b\xi_1}$	$c_{12} = \eta_1$
$c_{13} = \frac{1}{2}\left(1 + \sqrt{(1+2\kappa)^2 + 4b\xi_1}\right)$	$A = (2+b)\xi_1 - \widetilde{E}_{n\kappa}$
$B = -2\left(\widetilde{E}_{n\kappa} - \xi_1\right) - \kappa(\kappa+1)$	$C = -\widetilde{E}_{n\kappa}$

where $\tau'(z) < 0$ must be satisfied in order to obtain a physical solution according to the NU method.[38] Thirdly, we need to calculate the energy eigenvalues by means of the eigenvalue equation, relation A6, and obtain

$$\varepsilon_{n\kappa} = \frac{(2+b)\,b\nu_1^2}{2n+1+\sqrt{(1+2\kappa)^2+4b^2\nu_1^2}} - \frac{2n+1+\sqrt{(1+2\kappa)^2+4b^2\nu_1^2}}{4}. \tag{26}$$

Finally, with the aid of Eqs. (14) and (18), Eq. (26) can be also reduced to the energy equation for the GMP with the spin symmetry concept for any spin-orbit quantum number $\kappa = \pm 1, \pm 2, \ldots$ values in the Dirac theory,

$$\left(Mc^2 + E_{n\kappa} - C_s\right)\left(Mc^2 + D - E_{n\kappa}\right) + \hbar^2 c^2 \alpha^2 \kappa(\kappa+1)d_0$$

$$= \hbar^2 c^2 \alpha^2 \left(\frac{(2+b)\,b\nu_1^2}{2(n+\delta_1)} - \frac{(n+\delta_1)}{2}\right)^2, \ n = 0, 1, 2, \ldots, \tag{27}$$

where

$$\delta_1 = \frac{1}{2}\left(1 + \sqrt{(1+2\kappa)^2 + 4b^2\nu_1^2}\right) \geq 1. \tag{28}$$

In what follows, in order to establish the wave functions $F_{n\kappa}(r)$ of Eq. (8), the relations (A7–A10) are used. Firstly, we find the first part of the wave functions yields

$$\phi(z) = z^{\varepsilon_{n\kappa}}(1-z)^{\delta_1}, \ \varepsilon_{n\kappa} > 0, \ \delta_1 > 0. \tag{29}$$

Secondly, we calculate the weight function as

$$\rho(z) = z^{2\varepsilon_{n\kappa}}(1-z)^{2\delta_1 - 1} \tag{30}$$

and this, in turn, generates the second part of the wave functions,

$$y_n(z) \sim z^{-2\varepsilon_{n\kappa}}(1-z)^{-(2\delta_1-1)}\frac{d^n}{dz^n}\left[z^{n+2\varepsilon_{n\kappa}}(1-z)^{n+2\delta_1-1}\right] \approx P_n^{(2\varepsilon_{n\kappa},2\delta_1-1)}(1-2z), \tag{31}$$

where $P_n^{(a,b)}(1-2z)$ is the orthogonal Jacobi polynomials.[49,50] Finally, the upper spinor component $F_{n\kappa}(z)$ for arbitrary κ can be obtained by means of Eq. (20) as

$$F_{n\kappa}(r) = \mathcal{N}_{n\kappa}e^{-\varepsilon_{n\kappa}\alpha r}(1-e^{-\alpha r})^{\delta_1}P_n^{(2\varepsilon_{n\kappa},2\delta_1-1)}(1-2e^{-\alpha r})$$

$$= \mathcal{N}_{n\kappa}\frac{\Gamma(n+2\varepsilon_{n\kappa}+1)}{\Gamma(2\varepsilon_{n\kappa}+1)n!}e^{-\varepsilon_{n\kappa}\alpha r}(1-e^{-\alpha r})^{\delta_1}\,_2F_1\left(-n, n+2\varepsilon_{n\kappa}+2\delta_1; 1+2\varepsilon_{n\kappa}; e^{-\alpha r}\right), \tag{32}$$

where the normalization constants $\mathcal{N}_{n\kappa}$ are calculated in Appendix C.

Let us recall the derivative relation of the hypergeometric function,

$$\frac{d}{dz}\left[{}_2F_1\left(a;b;c;z\right)\right] = \left(\frac{ab}{c}\right){}_2F_1\left(a+1;b+1;c+1;z\right),$$

that is used to calculate the corresponding lower-component $G_{n\kappa}(r)$ in Eq. (9) as

$$G_{n\kappa}(r) = \frac{\mathcal{N}_{n\kappa}}{Mc^2 + E_{n\kappa} - C_s}\left(\frac{\alpha\delta_1 e^{-\alpha r}}{1 - e^{-\alpha r}} - \alpha\varepsilon_{n\kappa} + \frac{\kappa}{r}\right)F_{n\kappa}(r)$$

$$+ \mathcal{N}_{n\kappa}\frac{n\alpha\left(n + 2\varepsilon_{n\kappa} + 2\delta_1\right)}{\left(Mc^2 + E_{n\kappa} - C_s\right)\left(1 + 2\varepsilon_{n\kappa}\right)}(1 - e^{-\alpha r})^{\delta_1}\left(e^{-\alpha r}\right)^{\varepsilon_{n\kappa}+1}$$

$$\times {}_2F_1\left(1 - n; n + 2\left(\varepsilon_{n\kappa} + \delta_1\right) + 1; 2\left(1 + \varepsilon_{n\kappa}\right); e^{-\alpha r}\right). \tag{33}$$

Noting that the hypergeometric series $c_2F_1\left(1 - n; n + 2\left(\varepsilon_{n\kappa} + \delta_1\right) + 1; 2\left(1 + \varepsilon_{n\kappa}\right); e^{-\alpha r}\right)$ terminates for $n = 0$ and thus does not diverge for all values of real parameters $\varepsilon_{n\kappa}$ and δ_1.

In the presence of exact spin symmetry ($C_s = 0$), $E_{n\kappa} \neq -Mc^2$ (only positive energy states do exist). In the exact spin symmetry ($C_s = 0$) with ($\kappa = 1, \kappa = -2$), the upper- and lower-spinor wave functions for the ground ($0p_{1/2}, 0p_{3/2}$) and first excited ($1p_{1/2}, 1p_{3/2}$) degenerate eigenstates are being illustrated in Figs. 2(a) and 2(b), respectively. A glance at Fig. 2 reveals that there is only one curve (dashed line) for the radial wavefunctions of the upper components for both states in the doublet. However, there are two curves (solid lines) for the radial wavefunctions of the lower components. The following values of the parameters $M = 1.0\ fm^{-1}$, $D = 15\ fm^{-1}$, $\alpha = 0.1\ fm^{-1}$ and $r_e = 0.4\ fm$, $E_{0,\kappa=1} = E_{0,\kappa=-2} = 5.5791076\ fm^{-1}$ and $E_{1,\kappa=1} = E_{1,\kappa=-2} = 8.1823677\ fm^{-1}$ have been used.

Let us now study the special case when the parameter $\alpha \to 0$ in Eq. (7a). The GMP potential can be easily reduced to the well-known Kratzer molecular potential,

$$\lim_{\alpha\to 0} V(r) = D\left(\frac{r - r_e}{r}\right)^2, \tag{34}$$

which has been studied extensively by using different methods as the function analysis,[51] the NU,[52,53] and the exact quantization rule (EQR).[54] To avoid the repetition, following Appendix A, we write down the essential polynomials:

$$\pi(r) = \frac{1}{2}\left[1 + \gamma - 2\epsilon_{n\kappa}r\right], \tag{35}$$

$$k = \frac{2r_e}{\hbar^2c^2}\left(Mc^2 + E_{n\kappa} - C_s\right)D - \gamma\epsilon_{n\kappa}, \tag{36}$$

$$\tau(r) = 1 + \gamma - 2\epsilon_{n\kappa}r, \tag{37}$$

where

$$\gamma = \sqrt{(1 + 2\kappa)^2 + \frac{4r_e^2}{\hbar^2c^2}\left(Mc^2 + E_{n\kappa} - C_s\right)D}, \tag{38}$$

$$\epsilon_{n\kappa} = \frac{1}{\hbar c}\sqrt{\left(Mc^2 - E_{n\kappa} + D\right)\left(Mc^2 + E_{n\kappa} - C_s\right)}. \tag{39}$$

We further obtain the following two expressions which are relevant in the construction of the energy equation (relation A6)[38]

$$\lambda_n = 2n\epsilon_{n\kappa} \quad \text{and} \quad \lambda = \frac{2r_e}{\hbar^2c^2}\left(Mc^2 + E_{n\kappa} - C_s\right)D - (1 + \gamma)\epsilon_{n\kappa}, \tag{40}$$
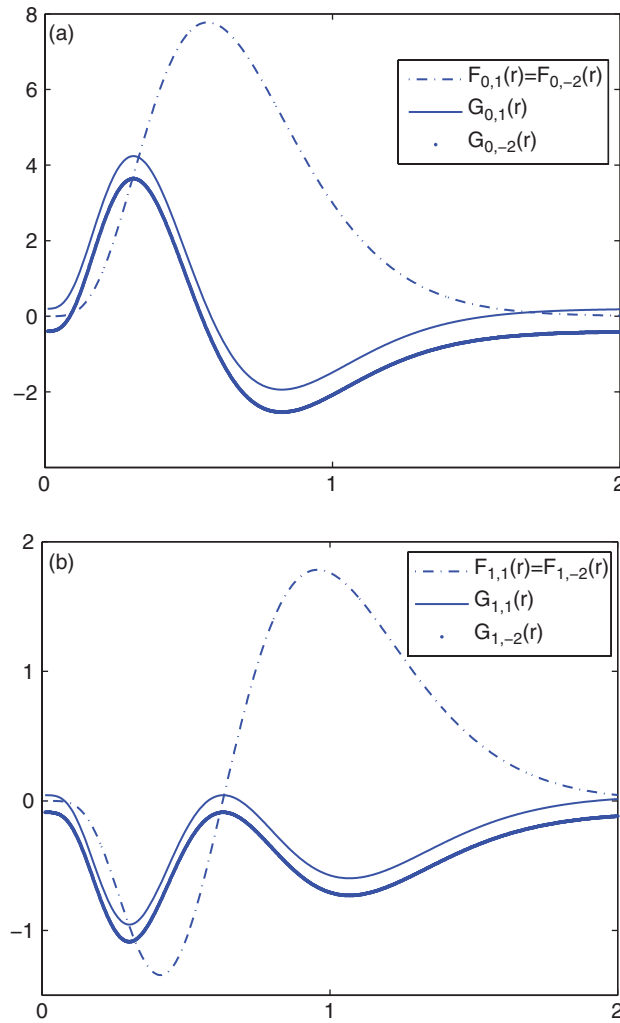
FIG. 2. (Color online) The upper and lower spinor wave functions, in the exact spin symmetry, for (a) $n = 0\,(0p_{1/2}, 0p_{3/2})$ spin doublet eigenstates with $k = 1$ and $k = -2$ and (b) $n = 1\,(1p_{1/2}, 1p_{3/2})$ spin doublet eigenstates with $k = 1$ and $k = -2$.

and substituting $\lambda_n = \lambda$, we can obtain the following spin symmetric energy equation,

$$\left(Mc^2 - E_{n\kappa} + D\right) = \frac{qD^2\left(Mc^2 + E_{n\kappa} - C_s\right)}{\left(N_n + \sqrt{N_\kappa^2 + qD\left(Mc^2 + E_{n\kappa} - C_s\right)}\right)^2}, \tag{41a}$$

where $N_n = 2n + 1$, $N_\kappa = 2\kappa + 1$, and $q = (2r_e/\hbar c)^2$. The above equation for energies looks like a quartic equation of the form:

$$a_4 E_{n\kappa}^4 + a_3 E_{n\kappa}^3 + a_2 E_{n\kappa}^2 + a_1 E_{n\kappa} + a_0 = 0 \tag{41b}$$

with coefficients

$$a_4 = q^2 D^2; \quad a_3 = 2qD\left[N_\kappa^2 - N_n^2 - qDC_s\right];$$

$$a_2 = (N_n^2 - N_\kappa^2)^2 + 2qD(D + M + C_s)(N_n^2 - N_\kappa^2) + 4qD^2 N_n^2 + q^2 D^2\left(C_s^2 + 2MC_s - 2M^2\right);$$

$$a_1 = 2q^2 D^2 M C_s(M - C_s) - 2(D + M)(N_n^2 - N_\kappa^2)^2$$
$$+ 2qD\left(M^2 - 2MC_s - DC_s\right)(N_n^2 - N_\kappa^2) - 4qD^2(D + C_s)N_n^2;$$

$$a_0 = \left\lceil qDM^2 - (D+M)(N_n^2 - N_\kappa^2)\right\rceil^2 - 4qD^2(M-C_s)(D+M)N_n^2$$
$$+ q^2D^2M^2C_s(C_s - 2M) + 2qDMC_s(D+M)(N_n^2 - N_\kappa^2);$$

where we have set $c=1$. For a given value of n and κ (or l), the above quartic equation, Eq. (7b), provides four distinct positive and negative real (real and complex) energy spectra related with $E_{n\kappa}^+$ or $E_{n\kappa}^-$, respectively. One of the distinct solutions is only valid to obtain the positive-energy bound states in the limit of the spin symmetry. Therefore, the procedures for calculating the four distinct energies; namely, $E_{n\kappa}^{(1)}$, $E_{n\kappa}^{(2)}$, $E_{n\kappa}^{(3)}$, and $E_{n\kappa}^{(4)}$ are given in Appendix B.

Furthermore, following Ref. 18 the normalized upper- and lower-spinor wave functions can be calculated as

$$F_{n\kappa}(r) = (2\epsilon_{n\kappa})^{K+1}\sqrt{\frac{\epsilon_{n\kappa}n!}{(n+K+1)\Gamma(n+2K+2)}}r^{K+1}e^{-\epsilon_{n\kappa}r}L_n^{(2K+1)}(2\epsilon_{n\kappa}r), \qquad (42)$$

and

$$G_{n\kappa}(r) = \frac{1}{Mc^2 + E_{n\kappa} - C_s}(2\epsilon_{n\kappa})^{K+1}\sqrt{\frac{\epsilon_{n\kappa}n!}{(n+K+1)\Gamma(n+2K+2)}}$$
$$\times\left[\left(\frac{(K+1)+\kappa}{r} - \epsilon_{n\kappa}\right)F_{n\kappa}(r) - 2\epsilon_{n\kappa}r^{K+1}e^{-\epsilon_{n\kappa}r}L_{n-1}^{(2K+2)}(2\epsilon_{n\kappa}r)\right],$$

$$K = \frac{1}{2}(\gamma - 1), \qquad (43)$$

respectively, where $L_n^{(\beta)}(x)$ are associated Laguerre polynomials. The simplest exact spin solution, representing the ground state and first excited state, are

$$\begin{cases} \left(Mc^2 + E_{0\kappa}\right)\left(Mc^2 - E_{0\kappa} + D\right) = \frac{1}{\hbar^2c^2}\left(\frac{r_e(Mc^2+E_{0\kappa})D}{1+\sqrt{(1+2\kappa)^2 + \frac{4Dr_e^2}{\hbar^2c^2}(Mc^2+E_{0\kappa})}}\right)^2, \\ F_{0\kappa}(r) = (2\epsilon_{0\kappa})^{K_0+1}\sqrt{\frac{\epsilon_{0\kappa}}{(K_0+1)\Gamma(2K_0+2)}}r^{K_0+1}e^{-\epsilon_{n\kappa}r}, \\ G_{0\kappa}(r) = \frac{1}{Mc^2+E_{0\kappa}}(2\epsilon_{0\kappa})^{K_0+1}\sqrt{\frac{\epsilon_{0\kappa}}{(K_0+1)\Gamma(2K_0+2)}}\left(\frac{(K_0+1)+\kappa}{r} - \epsilon_{0\kappa}\right)F_{0\kappa}(r), \end{cases}$$

$$\begin{cases} \left(Mc^2 + E_{1\kappa}\right)\left(D + Mc^2 - E_{1\kappa}\right) = \frac{1}{\hbar^2c^2}\left(\frac{Dr_e(Mc^2+E_{1\kappa})}{3+\sqrt{(1+2\kappa)^2 + \frac{4Dr_e^2}{\hbar^2c^2}(Mc^2+E_{1\kappa})}}\right)^2, \\ F_{1\kappa}(r) = (2\epsilon_{1\kappa})^{K_1+1}\sqrt{\frac{\epsilon_{1\kappa}}{(K_1+2)\Gamma(2K_1+3)}}r^{K_1+1}e^{-\epsilon_{1\kappa}r}(-2\epsilon_{1\kappa}r + 2K_1 + 2), \\ G_{1\kappa}(r) = \frac{1}{Mc^2+E_{1\kappa}}(2\epsilon_{1\kappa})^{K_1+1}\sqrt{\frac{\epsilon_{1\kappa}}{(K+2)\Gamma(2K+3)}}\left[\left(\frac{(K_1+1)+\kappa}{r} - \epsilon_{1\kappa}\right)F_{1\kappa}(r) - 2\epsilon_{1\kappa}r^{K_1+1}e^{-\epsilon_{1\kappa}r}\right], \end{cases}$$

respectively. Finally, we would like to note that $\epsilon_{0\kappa}$ and $\epsilon_{1\kappa}$ can be calculated via Eq. (39) whereas K_0 and K_1 via Eq. (43) along with Eq. (38) when $C_s = 0$.

B. Pseudospin symmetry solutions of the GMP model

From Eq. (10), we can see that the energy eigenvalues, $E_{n\kappa}$, depend only on n and $\tilde{l}$, i.e., $E_{n\kappa} = E(n, \tilde{l}(\tilde{l}+1))$. For $\tilde{l} \neq 0$, the states with $j = \tilde{l} \pm 1/2$ are degenerate. This is a $SU(2)$ pseudospin symmetry. Following Refs. 22–25, we impose the GMP (Ref. 1) as the $\Delta(r)$, i.e.,

$$\Delta(r) = D\left(1 - \frac{b}{e^{\alpha r} - 1}\right)^2, \qquad (44)$$

TABLE II. Specific values of the constants in the solution of the GMP under pseudospin symmetry.

Constant	Constant
$c_1 = 1$	$c_2 = 1$
$c_3 = 1$	$c_4 = 0$
$c_5 = -\frac{1}{2}$	$c_6 = \frac{1}{4}\left[1 + 4\left((2+b)\,\xi_2 - \widetilde{E}_{n\kappa}\right)\right]$
$c_7 = 2\left(\widetilde{E}_{n\kappa} - \xi_2\right) + \kappa\,(\kappa - 1)$	$c_8 = \eta_2^2 = -\widetilde{E}_{n\kappa}$
$c_9 = \frac{1}{4}\left[(1 - 2\kappa)^2 + 4b\xi_2\right]$	$c_{10} = 2\eta_2 = 2i\sqrt{\widetilde{E}_{n\kappa}}$
$c_{11} = \sqrt{(1 - 2\kappa)^2 + 4b\xi_2}$	$c_{12} = \eta_2$
$c_{13} = \frac{1}{2}\left(1 + \sqrt{(1 - 2\kappa)^2 + 4b\xi_2}\right)$	$A = (2+b)\,\xi_2 - \widetilde{E}_{n\kappa}$
$B = -2\left(\widetilde{E}_{n\kappa} - \xi_2\right) - \kappa\,(\kappa - 1)$	$C = \eta_2^2 = -\widetilde{E}_{n\kappa}$

leads us to obtain a Schrödinger-like equation in the form:

$$\left[\frac{d^2}{dr^2} - \frac{\kappa\,(\kappa - 1)}{r^2} + \alpha^2 v_2^2 \left(1 - \frac{be^{-\alpha r}}{1 - e^{-\alpha r}}\right)^2 + \alpha^2 \omega_2^2\right] G_{n\kappa}(r) = 0 \tag{45}$$

where

$$v_2^2 = \frac{1}{\alpha^2 \hbar^2 c^2}\left(Mc^2 - E_{n\kappa} + C_{ps}\right) D, \quad \omega_2^2 = \frac{1}{\alpha^2 \hbar^2 c^2}\left[E_{n\kappa}^2 - M^2 c^4 - C_{ps}\left(Mc^2 + E_{n\kappa}\right)\right] \tag{46}$$

where $\kappa\,(\kappa - 1) = \tilde{l}(\tilde{l} + 1)$. We follow the same procedures in the previous subsection to obtain a Dirac equation satisfying $G_{n\kappa}(r)$,

$$\left(\frac{d^2}{dz^2} + \frac{(1 - z)}{z(1 - z)}\frac{d}{dz}\right) G_{n\kappa}(z)$$

$$+ \frac{1}{z^2(1 - z)^2}\left\{-\left[\widetilde{\varepsilon}_{n\kappa}^2 - (2 + b)\,bv_2^2\right] z^2 + \left[2\left(\widetilde{\varepsilon}_{n\kappa}^2 - bv_2^2\right) - \kappa\,(\kappa - 1)\right] z - \widetilde{\varepsilon}_{n\kappa}^2\right\} G_{n\kappa}(z) = 0, \tag{47}$$

where

$$\widetilde{\varepsilon}_{n\kappa}^2 = -\omega_2^2 - v_2^2 + \kappa\,(\kappa - 1)\,d_0, \tag{48}$$

where $G_{n\kappa}(r) = G_{n\kappa}(z)$. To avoid repetition in the solution of Eq. (45), a careful inspection for the relationship between the present set of parameters $(\widetilde{\varepsilon}_{n\kappa}, v_2^2)$ and the previous one $(\varepsilon_{n\kappa}, v_1^2)$ tells us that the negative energy solution for pseudospin symmetry, where $S(r) \sim -V(r)$, can be obtained directly from the spin symmetric solution by using the following parameter mapping:[18,55]

$$F_{n\kappa}(r) \leftrightarrow G_{n\kappa}(r), \kappa \to \kappa - 1, \ V(r) \to -V(r) \ (\text{i.e., } D \to -D), \ E_{n\kappa} \to -E_{n\kappa} \text{ and } C_s \to -C_{ps}. \tag{49}$$

Following the previous procedures, the constants in the case of pseudospin symmetry concept are displayed in Table II. Applying the above transformations to Eqs. (27), (32), and (33) leads to the following pseudospin symmetric energy equation:

$$\left(E_{n\kappa} - Mc^2 - C_{ps}\right)\left(D - Mc^2 - E_{n\kappa}\right) + \hbar^2 c^2 \alpha^2 \kappa\,(\kappa - 1)\,d_0$$

$$= \hbar^2 c^2 \alpha^2 \left(\frac{(2 + b)\,bv_2^2}{2\,(n + \delta_2)} + \frac{(n + \delta_2)}{2}\right)^2, \ n = 0, 1, 2, \ldots, \tag{50}$$

with

$$\delta_2 = \frac{1}{2}\left(1 + \sqrt{(1 - 2\kappa)^2 - 4b^2 v_2^2}\right) \geq 1. \tag{51}$$

Note that Eq. (50) can be also expressed in the form of quartic equation (cf. Eq. (7b)) since the two parameters v_2^2 and δ_2 contain the energy eigenvalues $E_{n\kappa}$. The procedures of this analytic solution

is so similar to the one presented in Appendix B with the changes $D \to -D$, $E_{n\kappa} \to -E_{n\kappa}$, and $C_s \to -C_{ps}$. Furthermore, the lower-component wave functions:

$$G_{n\kappa}(r) = \mathcal{N}_{n\kappa} e^{-\widetilde{\varepsilon}_{n\kappa}\alpha r}(1 - e^{-\alpha r})^{\delta_2} P_n^{(2\widetilde{\varepsilon}_{n\kappa}, 2\delta_2 - 1)}(1 - 2e^{-\alpha r})$$

$$= \mathcal{N}_{n\kappa} \frac{(2\widetilde{\varepsilon}_{n\kappa} + 1)_n}{n!} e^{-\widetilde{\varepsilon}_{n\kappa}\alpha r}(1 - e^{-\alpha r})^{\delta_2} {}_2F_1\left(-n, n + 2\widetilde{\varepsilon}_{n\kappa} + 2\delta_2; 1 + 2\widetilde{\varepsilon}_{n\kappa}; e^{-\alpha r}\right), \qquad (52)$$

where

$$\widetilde{\varepsilon}_{n\kappa} = -\left(\frac{(2 + b)\, b v_2^2}{2n + 1 + \sqrt{(1 - 2\kappa)^2 - 4b^2 v_2^2}} + \frac{2n + 1 + \sqrt{(1 - 2\kappa)^2 - 4b^2 v_2^2}}{4}\right). \qquad (53)$$

The upper-component $F_{n\kappa}(r)$ can be calculated from Eq. (11) as follows

$$F_{n\kappa}(r) = \frac{\mathcal{N}_{n\kappa}}{Mc^2 - E_{n\kappa} + C_{ps}}\left(\frac{\alpha\delta_2 e^{-\alpha r}}{1 - e^{-\alpha r}} - \alpha\widetilde{\varepsilon}_{n\kappa} - \frac{\kappa}{r}\right) G_{n\kappa}(r)$$

$$+ \mathcal{N}_{n\kappa} \frac{n\alpha\, (n + 2\widetilde{\varepsilon}_{n\kappa} + 2\delta_2)}{(Mc^2 - E_{n\kappa} + C_{ps})\,(1 + 2\widetilde{\varepsilon}_{n\kappa})}(1 - e^{-\alpha r})^{\delta_2}\left(e^{-\alpha r}\right)^{\widetilde{\varepsilon}_{n\kappa} + 1}$$

$$\times {}_2F_1\left(1 - n; n + 2\left(\widetilde{\varepsilon}_{n\kappa} + \delta_2\right) + 1; 2\left(1 + \widetilde{\varepsilon}_{n\kappa}\right); e^{-\alpha r}\right). \qquad (54)$$

Hence, in the exact pseudospin symmetry where $C_{ps} = 0$, $E_{n\kappa} \neq Mc^2$ (only negative energy states exist).

On the other hand, the pseudospin solutions of the Dirac equation for the Kratzer potential can be obtained from the spin symmetry case by applying transformation map given by Eq. (49) as

$$\widetilde{\gamma} = \sqrt{(1 - 2\kappa)^2 - \frac{4Dr_e^2}{\hbar^2 c^2}\left(Mc^2 - E_{n\kappa} + C_{ps}\right)}, \qquad (55)$$

$$\widetilde{\epsilon}_{n\kappa} = \frac{1}{\hbar c}\sqrt{-D\left(Mc^2 - E_{n\kappa} + C_{ps}\right) - \left(E_{n\kappa}^2 - M^2 c^4 - C_{ps}\left(Mc^2 + E_{n\kappa}\right)\right)}. \qquad (56)$$

Therefore, the eigenvalue equation is

$$E_{n\kappa}^2 - M^2 c^4 - C_{ps}\left(Mc^2 + E_{n\kappa}\right) = -D\left(Mc^2 - E_{n\kappa} + C_{ps}\right)$$

$$- \frac{1}{\hbar^2 c^2}\left(\frac{Dr_e\left(Mc^2 - E_{n\kappa} + C_{ps}\right)}{n + K + 1}\right)^2, \quad \widetilde{K} = \frac{1}{2}\left(\widetilde{\gamma} - 1\right), \qquad (57)$$

and the normalized lower- and upper-spinor wave functions are given by

$$G_{n\kappa}(r) = (2\widetilde{\epsilon}_{n\kappa})^{\widetilde{K}+1}\sqrt{\frac{\widetilde{\epsilon}_{n\kappa} n!}{\left(n + \widetilde{K} + 1\right)\Gamma\left(n + 2\widetilde{K} + 2\right)}}\, r^{\widetilde{K}+1} e^{-\widetilde{\epsilon}_{n\kappa} r} L_n^{(2\widetilde{K}+1)}(2\widetilde{\epsilon}_{n\kappa} r), \qquad (58)$$

$$F_{n\kappa}(r) = \frac{1}{Mc^2 - E_{n\kappa} + C_{ps}}(2\widetilde{\epsilon}_{n\kappa})^{\widetilde{K}+1}\sqrt{\frac{\widetilde{\epsilon}_{n\kappa} n!}{\left(n + \widetilde{K} + 1\right)\Gamma\left(n + 2\widetilde{K} + 2\right)}}$$

$$\times \left[\left(\frac{(\widetilde{K} + 1) - \kappa}{r} - \widetilde{\epsilon}_{n\kappa}\right) G_{n\kappa}(r) - 2\widetilde{\epsilon}_{n\kappa} r^{\widetilde{K}+1} e^{-\widetilde{\epsilon}_{n\kappa} r} L_{n-1}^{(2\widetilde{K}+2)}(2\widetilde{\epsilon}_{n\kappa} r)\right]. \qquad (59)$$

III. SOME SPECIAL CASES

Let us study three special cases. At first, we study the $s(\widetilde{s})$ -states ($l = \widetilde{l} = 0$, i.e., $\kappa = \mp 1$). It follows that the spin-orbit coupling term $\kappa(\kappa + 1)/r^2 = 0$, and also the corresponding approximation

to it in Eq. (27). In the presence of the exact spin symmetry limit ($C_s = 0$), the s-states ($\kappa = -1$) energy equation becomes

$$E_{n,-1}^2 - M^2c^4 - D\left(Mc^2 + E_{n,-1}\right) = -\hbar^2c^2\alpha^2 \left(\frac{\frac{1}{\hbar^2c^2\alpha^2}\left(2+b\right)\left(Mc^2 + E_{n,-1}\right)Db}{2\left(n+\delta\right)} - \frac{n+\delta}{2}\right)^2,$$

(60)

where

$$\delta = \frac{1}{2}\left(1 + \sqrt{1 + \frac{4}{\hbar^2c^2\alpha^2}\left(Mc^2 + E_{n,-1}\right)Db^2}\right) \geq 1.$$

(61)

The upper- and lower-spinor wave functions are

$$F_{n,-1}(r) = \mathcal{N}_{n,-1}\frac{(2\eta+1)_n}{n!}e^{-\eta\alpha r}(1 - e^{-\alpha r})^\delta \, {}_2F_1\left(-n, n+2\eta+2\delta; 1+2\eta; e^{-\alpha r}\right),$$

(62)

and

$$G_{n,-1}(r) = \frac{\mathcal{N}_{n,-1}}{Mc^2 + E_{n,-1}}\left(\frac{\alpha\delta e^{-\alpha r}}{1 - e^{-\alpha r}} - \alpha\eta - \frac{1}{r}\right)F_{n,-1}(r)$$

$$+\mathcal{N}_{n,-1}\frac{n\alpha\left(n+2\eta+2\delta\right)}{\left(Mc^2 + E_{n,-1}\right)\left(1+2\eta\right)}(1 - e^{-\alpha r})^\delta \left(e^{-\alpha r}\right)^{\eta+1}$$

$$\times {}_2F_1\left(1-n; n+2\left(\eta+\delta\right)+1; 2\left(1+\eta\right); e^{-\alpha r}\right),$$

(63)

with

$$\eta = \frac{\frac{1}{\hbar^2c^2\alpha^2}\left(2+b\right)\left(Mc^2 + E_{n,-1}\right)Db}{2\left(n+\delta\right)} - \frac{n+\delta}{2},$$

(64)

where $\mathcal{N}_{n,-1}$ is calculated in the Appendix C. Moreover, for the $\widetilde{s}$-states ($\kappa = 1$) in the exact pseudospin symmetry ($C_{ps} = 0$), the energy equation (48) becomes

$$E_{n1}^2 - M^2c^4 + D\left(Mc^2 - E_{n1}\right) = -\hbar^2c^2\alpha^2 \left(\frac{\frac{1}{\hbar^2c^2\alpha^2}\left(2+b\right)\left(Mc^2 - E_{n1}\right)Db}{2\left(n+\delta_2\right)} + \frac{(n+\delta_2)}{2}\right)^2,$$

(65)

with

$$\delta_2 = \frac{1}{2}\left(1 + \sqrt{1 - \frac{4}{\alpha^2\hbar^2c^2}\left(mc^2 - E_{n1}\right)Db^2}\right), \quad E_{n1} > mc^2, \; n = 0, 1, 2, \ldots.$$

(66)

The wave functions given by Eqs. (52) and (54) become

$$G_{n1}(r) = \mathcal{N}_{n1}e^{-\eta_2\alpha r}(1 - e^{-\alpha r})^{\delta_2}P_n^{(2\eta_2, 2\delta_2-1)}(1 - 2e^{-\alpha r})$$

$$= \mathcal{N}_{n1}\frac{(2\eta_2+1)_n}{n!}e^{-\eta_2\alpha r}(1 - e^{-\alpha r})^{\delta_2} \, {}_2F_1\left(-n, n+2\eta_2+2\delta_2; 1+2\eta_2; e^{-\alpha r}\right),$$

(67)

with

$$\eta_2 = \frac{(2+b)\xi_2}{2n+1+\sqrt{1+4b\xi_2}} - \frac{2n+1+\sqrt{1+4b\xi_2}}{4},$$

(68)

and

$$F_{n1}(r) = \frac{\mathcal{N}_{n1}}{Mc^2 - E_{n1}}\left(\frac{\alpha\delta_2 e^{-\alpha r}}{1 - e^{-\alpha r}} - \alpha\eta_2 - \frac{1}{r}\right)G_{n\kappa}(r)$$

$$+\mathcal{N}_{n1}\frac{n\alpha\left(n+2\eta_2+2\delta_2\right)}{\left(Mc^2 - E_{n\kappa}\right)\left(1+2\eta_2\right)}(1 - e^{-\alpha r})^{\delta_2}\left(e^{-\alpha r}\right)^{\eta_2+1}$$

$$\times {}_2F_1\left(1-n; n+2\left(\eta_2+\delta_2\right)+1; 2\left(1+\eta_2\right); e^{-\alpha r}\right),$$

(69)

where $E_{n1} \neq Mc^2$.

Second, we study the nonrelativistic limit. In applying the following appropriate mapping $E_{n\kappa} - Mc^2 \rightarrow E_{nl}$ and $\frac{1}{\hbar^2 c^2}\left(Mc^2 + E_{n\kappa}\right) \rightarrow \frac{2\mu}{\hbar^2}$ to Eqs. (27) and (32), we obtain the energy levels of the Schrödinger equation for any arbitrary orbital quantum number l as

$$E_{nl} = D + \frac{\hbar^2}{2\mu}l(l+1)\alpha^2 d_0 - \frac{\hbar^2\alpha^2}{2\mu}\left(\frac{\frac{\mu}{\hbar^2\alpha^2}(2+b)Db}{n+\widetilde{\delta}} - \frac{n+\widetilde{\delta}}{2}\right)^2, \qquad (70)$$

and the radial wave functions as

$$\psi_{nl}(r) = \mathcal{N}_{nl}\frac{(2\widetilde{\eta}+1)_n}{n!}e^{-\widetilde{\eta}\alpha r}(1-e^{-\alpha r})^{\widetilde{\delta}}{}_2F_1\left(-n, n+2\widetilde{\eta}+2\widetilde{\delta}; 1+2\widetilde{\eta}; e^{-\alpha r}\right), \qquad (71)$$

with

$$\widetilde{\delta} = \frac{1}{2}\left(1+\sqrt{(1+2l)^2 + \frac{8\mu}{\hbar^2\alpha^2}Db^2}\right) \geq 1, \quad \widetilde{\eta} = \frac{\frac{\mu}{\hbar^2\alpha^2}(2+b)Db}{n+\widetilde{\delta}} - \frac{n+\widetilde{\delta}}{2}, \qquad (72)$$

where $\mu = m_1 m_2/(m_1 + m_2)$ is the reduced mass of the two atoms and $\mathcal{N}_{nl}$ is calculated in the Appendix C.

Third, the case $\alpha \rightarrow 0$, the nonrelativistic limit of the bound state solutions of Eqs. (41) and (42) in exact symmetry limit ($C_s = 0$) can be obtained as

$$E_{nl} = D - -\frac{8\mu}{\hbar^2}\left(\frac{Dr_e}{1+2n+\sqrt{(1+2l)^2 + \frac{8\mu}{\hbar^2}Dr_e^2}}\right)^2, \qquad (73)$$

and the normalized wave functions:

$$\psi_{nl}(r) = (2\widetilde{\epsilon}_{nl})^{L+1}\sqrt{\frac{\widetilde{\epsilon}_{nl}n!}{(n+L+1)\Gamma(n+2L+2)}}r^{L+1}e^{-\widetilde{\epsilon}_{nl}r}L_n^{(2L+1)}(2\widetilde{\epsilon}_{nl}r), \qquad (74)$$

where

$$\widetilde{\epsilon}_{nl} = \frac{2\mu Dr_e}{\hbar^2(n+L+1)} \quad \text{and} \quad L = \frac{1}{2}\left(\sqrt{(1+2l)^2 + \frac{8\mu}{\hbar^2}Dr_e^2} - 1\right), \qquad (75)$$

which are identical to the previous results obtained by the function analysis method,[51] the NU method,[52,53] and the EQR (cf. Eq. (29) in Ref. 54 obtained in D-dimensions).

IV. NUMERICAL RESULTS

As in Ref. 10, we calculate the non-relativistic energy levels as function of various values of the parameter that controls the width of the potential well $\alpha = 0.05$–$0.30\ fm^{-1}$ and equilibrium internuclear distance $r_e = 0.4, 0.8\ fm$ for various states with quantum numbers n and l. The atomic units $\hbar = M = 1$ are used and the dissociation energy is set to $D = 15\ fm^{-1}$. In Table III, we display our results with those ones calculated by using the conventional approximation scheme suggested by Greene and Aldrich[12] to deal with the centrifugal term $l(l+1)/r^2$ together with the values obtained from the numerical integration procedures based on the MATHEMATICA package programmed by Lucha and Schöberl.[56] Obviously, our results are closely approaching the ones obtained in Ref. 56 for both short range (small α) as well as for long range (large α) potential (see, e.g., our recent works[39,57]). This means that our new approximated calculations using the approximation scheme (15) proposed recently by us provides much better approximation to the centrifugal term than that in Ref. 10 even for large α values (see, e.g., Ref. 40 and the references therein). At small values of α (Kratzer potential), we have also calculated the energy levels as a function of $r_e = 0.1 - 1.5\ fm$ for various quantum numbers n and l. Hence, our numerical values of these energy levels are shown in Table IV. Overmore, Table V presents some numerical values for the eigenenergies of the Dirac valence states obtained from Eq. (27) with parameters $M = 1.0\ fm^{-1}$, $D = 15\ fm^{-1}$, $\alpha = 0.1, 0.3\ fm^{-1}$, and $r_e = 0.4, 0.8\ fm$ (exact spin symmetry case, i.e., $C_s = 0\ fm^{-1}$). We noticed that there

TABLE III. The Schrödinger bound state energy levels E_{nl} (in au) of the GMP as functions of α and r_e for various states with $D = 15$ where $\hbar = \mu = 1$.

States	α	$r_e = 0.4$ Present	DG (Ref. 10)	LS (Ref. 56)	$r_e = 0.8$ Present	DG (Ref. 10)	LS (Ref. 56)
2p	0.05	7.86080	7.8606	7.8628	4.14088	4.14068	4.14208
	0.10	7.95329	7.95247	7.95537	4.21917	4.21835	4.2204
	0.15	8.04508	8.04322	8.04724	4.29737	4.29552	4.2987
	0.20	8.13616	8.13287	8.13842	4.37551	4.37221	4.3769
	0.25	8.22656	8.22142	8.22892	4.45360	4.44845	4.4551
	0.30	8.31630	8.30889	8.31874	4.53166	4.52425	4.5332
3p	0.05	10.9978	10.9976	10.9998	7.53279	7.53258	7.5350
	0.10	11.1626	11.1617	11.1647	7.72475	7.72393	7.7271
	0.15	11.3242	11.3224	11.32647	7.91516	7.9133	7.9177
	0.20	11.4828	11.4795	11.48513	8.10400	8.10071	8.1066
	0.25	11.6383	11.6331	11.64068	8.29129	8.28615	8.2941
	0.30	11.7907	11.7833	11.67565	8.47703	8.46962	8.4799
3d	0.05	10.2160	10.2154	10.21651	5.73974	5.73913	5.7404
	0.10	10.3535	10.351	10.35409	5.84574	5.84327	5.8465
	0.15	10.4893	10.4837	10.48992	5.95061	5.94505	5.9515
	0.20	10.6233	10.6135	10.62403	6.05441	6.04453	6.0553
	0.25	10.7557	10.7403	10.75645	6.15720	6.14177	6.1582
	0.30	10.8864	10.8642	10.88719	6.25904	6.23682	6.2601
4p	0.05	12.4976	12.4974	12.4992	9.61301	9.6128	9.6156
	0.10	12.6968	12.696	12.69851	9.88351	9.88269	9.8862
	0.15	12.8883	12.8865	12.8901	10.1485	10.1467	10.1514
	0.20	13.0722	13.0689	13.0740	10.4080	10.4047	10.4111
	0.25	13.2484	13.2433	13.2501	10.6619	10.6568	10.665
4d	0.05	12.0983	12.0977	12.0989	8.49334	8.49272	8.4948
	0.10	12.2850	12.2825	12.2857	8.70708	8.70461	8.7087
	0.15	12.4664	12.4608	12.46715	8.91774	8.91218	8.9194
	0.20	12.6424	12.6326	12.64324	9.12538	9.11551	9.1272
4f	0.05	11.8208	11.8195	11.8209	7.43469	7.43346	7.4351
	0.10	11.9979	11.993	11.9981	7.58636	7.58142	7.5868
	0.15	12.1716	12.1604	12.1718	7.73559	7.72448	7.7361
	0.20	12.3418	12.3221	12.3421	7.88251	7.86276	7.8831
5p	0.10	13.5421	13.5413	13.5434	11.3021	11.3012	11.3047
	0.20	13.9289	13.9257	13.9301	11.9132	11.9099	11.9161
5d	0.10	13.3068	13.3043	13.3075	10.5201	10.5176	10.5219
	0.20	13.6925	13.6827	13.6931	11.0692	11.0594	11.0713
5f	0.10	13.1475	13.1426	13.1478	9.7966	9.79166	9.7975
	0.20	13.5332	13.5134	13.5333	10.2728	10.253	10.2738
5g	0.10	13.0379	13.0296	13.0379	9.15212	9.14389	9.1524
	0.20	13.4267	13.3938	13.42667	9.55246	9.51954	9.5528
6p	0.10	14.0521	14.0513	14.0530	12.2798	12.279	12.2822
6d	0.10	13.9070	13.9045	13.9075	11.7364	11.7339	11.7383
6f	0.10	13.8111	13.8062	13.8113	11.2448	11.2398	11.2459
6g	0.10	13.7465	13.7383	13.7466	10.8152	10.807	10.8158

are only positive energy bound state solutions in the spin symmetry limit. One can also see from Table V that there are degeneracies between the eigenstates $(np_{1/2}, np_{3/2})$, $(nd_{3/2}, nd_{5/2})$, $(nf_{5/2}, nf_{7/2})$, $(ng_{7/2}, ng_{9/2})$, etc. In fact, each of these eigenstates form a spin doublet. For instance, for any specific value of n, where $n = 0, 1, 2, \ldots$, $np_{1/2}$ with $\kappa = 1$ is considered as the partner of $np_{3/2}$ with $\kappa = -2$. Thus, states that have the same radial n and orbital angular momentum l quantum numbers with $j = l + 1/2$ and $j = l - 1/2$ are degenerate.[58]

TABLE IV. The Schrödinger bound state energy levels E_{nl} (in au) of the Kratzer potential as a function of r_e for various states with $D = 15$ where $\hbar = \mu = 1$.

State/r_e =	0.1	0.4	0.8	1.0	1.5
2p	13.9765	7.76759	4.06249	3.21339	2.07749
3p	14.5308	10.8298	7.33925	6.26836	4.56776
3d	14.5192	10.0766	5.63256	4.41694	2.74032
4p	14.7319	12.2908	9.33707	8.27299	6.40188
4d	14.7269	11.9062	8.27643	7.04418	5.05045
4f	14.7246	11.6401	7.28048	5.81633	3.61468
5p	14.8268	13.0996	10.6443	9.65890	7.79162
5d	14.8242	12.8774	9.94062	8.80199	6.76420
5f	14.8230	12.7278	9.29975	7.96875	5.69315
5g	14.8224	12.6242	8.73653	7.18597	4.60125
6p	14.8789	13.5937	11.5460	10.6568	8.86976
6d	14.8774	13.4539	11.0555	10.0356	8.07050
6f	14.8768	13.3616	10.6191	9.44444	7.25037
6g	14.8764	13.2985	10.2430	8.89964	6.42694

TABLE V. The eigenenergies of the Dirac valence states in units of fm^{-1} for several values of n and κ with the parameters $M = 1.0\ fm^{-1}$ and $D = 15\ fm^{-1}$ in the case of exact spin symmetry limit ($C_s = 0\ fm^{-1}$). We have set $\hbar = c = 1$.

l	$n, \kappa < 0$	$nL_{j=l+1/2}$	$E_{n,\kappa<0}$	$n, \kappa > 0$	$nL_{j=l-1/2}$	$E_{n,\kappa>0}$
$\alpha = 0.10\ fm^{-1}$,	$r_e = 0.40\ fm$					
1	0, −2	$0p_{3/2}$	5.5791076	0, 1	$0p_{1/2}$	5.5791076
2	0, −3	$0d_{5/2}$	6.8118605	0, 2	$0d_{3/2}$	6.8118605
3	0, −4	$0f_{7/2}$	8.0171073	0, 3	$0f_{5/2}$	8.0171073
4	0, −5	$0g_{9/2}$	9.1025175	0, 4	$0g_{7/2}$	9.1025175
1	1, −2	$1p_{3/2}$	8.1823677	1, 1	$1p_{1/2}$	8.1823677
2	1, −3	$1d_{5/2}$	8.8815340	1, 2	$1d_{3/2}$	8.8815340
3	1, −4	$1f_{7/2}$	9.6603105	1, 3	$1f_{5/2}$	9.6603105
4	1, −5	$1g_{9/2}$	10.4200196	1, 4	$1g_{7/2}$	10.4200196
$\alpha = 0.30\ fm^{-1}$,	$r_e = 0.40\ fm$					
1	0, −2	$0p_{3/2}$	5.7078594	0, 1	$0p_{1/2}$	5.7078594
2	0, −3	$0d_{5/2}$	6.9646771	0, 2	$0d_{3/2}$	6.9646771
3	0, −4	$0f_{7/2}$	8.2121326	0, 3	$0f_{5/2}$	8.2121326
4	0, −5	$0g_{9/2}$	9.3506414	0, 4	$0g_{7/2}$	9.3506414
1	1, −2	$1p_{3/2}$	8.4626850	1, 1	$1p_{1/2}$	8.4626850
2	1, −3	$1d_{5/2}$	9.1762544	1, 2	$1d_{3/2}$	9.1762544
3	1, −4	$1f_{7/2}$	9.9831712	1, 3	$1f_{5/2}$	9.9831712
4	1, −5	$1g_{9/2}$	10.7812870	1, 4	$1g_{7/2}$	10.7812870
$\alpha = 0.10\ fm^{-1}$,	$r_e = 0.80\ fm$					
1	0, −2	$0p_{3/2}$	3.6831690	0, 1	$0p_{1/2}$	3.6831690
2	0, −3	$0d_{5/2}$	4.3378367	0, 2	$0d_{3/2}$	4.3378367
3	0, −4	$0f_{7/2}$	5.0775317	0, 3	$0f_{5/2}$	5.0775317
4	0, −5	$0g_{9/2}$	5.8291637	0, 4	$0g_{7/2}$	5.8291637
1	1, −2	$1p_{3/2}$	5.8388616	1, 1	$1p_{1/2}$	5.8388616
2	1, −3	$1d_{5/2}$	6.2180443	1, 2	$1d_{3/2}$	6.2180443
3	1, −4	$1f_{7/2}$	6.6999906	1, 3	$1f_{5/2}$	6.6999906
4	1, −5	$1g_{9/2}$	7.2334098	1, 4	$1g_{7/2}$	7.2334098

TABLE VI. The eigenenergies of the Dirac hole states in units of fm^{-1} for several values of n and κ with the parameters $M = 1.0\ fm^{-1}$ and $D = 15\ fm^{-1}$ in the case of the pseudospin limit. We have set $\hbar = c = 1$.

$\tilde{l}$	$n, \kappa < 0$	(l, j)	$E_{n,\kappa<0}$	$E_{n,\kappa<0}$	$n-1, \kappa > 0$	$(l+2, j+1)$	$E_{n-1,\kappa>0}$	$E_{n-1,\kappa>0}$
$\alpha = 0.10\ fm^{-1}$,	**$r_e = 0.40\ fm$**		$C_{ps} = 0$	$C_{ps} = 5.0$			$C_{ps} = 0$	$C_{ps} = 5.0$
1	1, −1	$1s_{1/2}$	7.1975980	9.0681299	0,2	$0d_{3/2}$	7.1975980	9.0681299
2	1, −2	$1p_{3/2}$	7.9184956	9.7118773	0,3	$0f_{5/2}$	7.9184956	9.7118773
3	1, −3	$1d_{5/2}$	8.6773995	10.3429382	0,4	$0g_{7/2}$	8.6773995	10.3429382
4	1, −4	$1f_{7/2}$	9.3887565	10.9088876	0,5	$0h_{9/2}$	9.3887565	10.9088876
1	2, −1	$2s_{1/2}$	8.7054710	10.1707692	1,2	$1d_{3/2}$	8.7054710	10.1707692
2	2, −2	$2p_{3/2}$	9.1768973	10.6114449	1,3	$1f_{5/2}$	9.1768973	10.6114449
3	2, −3	$2d_{5/2}$	9.7101038	11.0728353	1,4	$1g_{7/2}$	9.7101038	11.0728353
4	2, −4	$2f_{7/2}$	10.2352574	11.5024429	1,5	$1h_{9/2}$	10.2352574	11.5024429
$\alpha = 0.30\ fm^{-1}$,	**$r_e = 0.40\ fm$**		$C_{ps} = 0$	$C_{ps} = -5.0$			$C_{ps} = 0$	$C_{ps} = -5.0$
1	1, −1	$1s_{1/2}$	7.4924218	6.2513272	0,2	$0d_{3/2}$	7.4924218	6.2513272
2	1, −2	$1p_{3/2}$	8.2327097	6.9536060	0,3	$0f_{5/2}$	8.2327097	6.9536060
3	1, −3	$1d_{5/2}$	9.0263198	7.7574142	0,4	$0g_{7/2}$	9.0263198	7.7574142
4	1, −4	$1f_{7/2}$	9.7816289	8.5599024	0,5	$0h_{9/2}$	9.7816289	8.5599024
1	2, −1	$2s_{1/2}$	9.1118917	8.0615654	1,2	$1d_{3/2}$	9.1118917	8.0615654
2	2, −2	$2p_{3/2}$	9.5965089	8.5254559	1,3	$1f_{5/2}$	9.5965089	8.5254559
3	2, −3	$2d_{5/2}$	10.1539358	9.0879925	1,4	$1g_{7/2}$	10.1539358	9.0879925
4	2, −4	$2f_{7/2}$	10.7110039	9.6762585	1,5	$1h_{9/2}$	10.7110039	9.6762585
$\alpha = 0.10\ fm^{-1}$,	**$r_e = 0.80\ fm$**		$C_{ps} = 0$	$C_{ps} = -10.0$			$C_{ps} = 0$	$C_{ps} = -10.0$
1	1, −1	$1s_{1/2}$	4.9972498	2.8369925	0,2	$0d_{3/2}$	4.9972498	2.8369925
2	1, −2	$1p_{3/2}$	5.4281366	3.1294486	0,3	$0f_{5/2}$	5.4281366	3.1294486
3	1, −3	$1d_{5/2}$	5.9435202	3.5245459	0,4	$0g_{7/2}$	5.9435202	3.5245459
4	1, −4	$1f_{7/2}$	6.4876355	3.9879463	0,5	$0h_{9/2}$	6.4876355	3.9879463
1	2, −1	$2s_{1/2}$	6.3778620	4.3337279	1,2	$1d_{3/2}$	6.3778620	4.3337279
2	2, −2	$2p_{3/2}$	6.6705985	4.5479115	1,3	$1f_{5/2}$	6.6705985	4.5479115
3	2, −3	$2d_{5/2}$	7.0449439	4.8443318	1,4	$1g_{7/2}$	7.0449439	4.8443318
4	2, −4	$2f_{7/2}$	7.4613765	5.2011464	1,5	$1h_{9/2}$	7.4613765	5.2011464

On the other hand, in Table VI, we present some numerical values for the eigenenergies of the Dirac hole states obtained from Eq. (50) with the previous choice of potential parameters for the case of pseudospin symmetry limit ($C_{ps} = 0$, 5.0, -5.0 and $-10.0\ fm^{-1}$). One can see from Table VI that there are degeneracies between the eigenstates $(ns_{1/2}, (n-1)\,d_{3/2})$, $(np_{3/2}, (n-1)\,f_{5/2})$, $(nd_{5/2}, (n-1)\,g_{7/2})$, $(nf_{7/2}, (n-1)\,h_{9/2})$, etc. In fact, each of these eigenstates forms a pseudospin doublet. For instance, for specific value of $n = 1$, $1s_{1/2}$ with $\kappa = -1$ is considered as the partner of $0d_{3/2}$ with $\kappa = 2$. Thus, states that have pseudo orbital angular momentum $\tilde{l}$ quantum numbers, radial n and $n-1$ with $j = \tilde{l} - 1/2$ and $j = \tilde{l} + 1/2$, respectively, are degenerate.[58]

V. CONCLUSIONS

We have studied the approximate bound state solutions of the Dirac equation for the GMP model with any arbitrary spin-orbit κ state under the conditions of the spin (pseudospin) symmetry limitation by means of the NU method including a new improved approximation scheme to approximate the centrifugal (pseudo-centrifugal) barrier term. By setting $\Sigma(r)\,(\Delta(r))$ to the spherically symmetric GMP, we have derived the solutions of the Dirac equation for the relativistic energy eigenvalues and associated two-component spinor wave functions for arbitrary spin-orbit κ state that provides an approximate solution to the spin- and pseudo-spin symmetry. The resulting solutions of the wave functions are being expressed in terms of the Jacobi polynomials (or hypergeometric functions). We have shown that the present spin symmetry can be easily reduced to the non-relativistic solution once we set $V(r) = S(r)$ (i.e., $\Delta(r) = 0$ or $C_s = 0$). The non-relativistic limits of our solution are obtained by imposing appropriate transformations and recalling $\kappa(\kappa + 1) \to l(l+1)$ in the spin symmetry limits. Furthermore, when $\alpha \to 0$, our results can be reduced to the well-known bound state solutions for the Kratzer potential model. If we choose the spin-orbit quantum number $\kappa = -1$ ($\kappa = 1$) for spin (pseudospin) symmetry, the problem reduces to the exact $s(\tilde{s})$-wave Dirac solution. We must point out that the numerical calculations for eigenenergy of the Dirac states involved in Eqs. (27) and (50) are sensitive to the choice of the parameters C_s, C_{ps}, α, r_e, D and M. It is found that the spin (pseudospin) limit Dirac eigenenergy valence (hole) states along with the two-component spinors are identical with the results obtained previously by other methods and works.[15] It is noticed that these results are obtained in a much simpler fashion than Ref. 15. Finally, Eqs. (27) and (50) can be used to evaluate the binding energies of the GMP for diatomic molecules such as CH, CO, and N_2 (Refs. 2 and 10) in the relativistic framework with spin and pseudospin symmetry cases for any range of potential well α. Equations (41) and (57) can be also used to evaluate the binding energy of the Kratzer potential in the relativistic framework with spin and pseudospin symmetry cases at small range potential well ($\alpha \to 0$).

APPENDIX A: GENERALIZATION OF THE NU METHOD

We briefly review the Nikiforov-Uvarov essential polynomial functions, root, eigenvalues, and wave functions (see Eqs. (5)–(11) of Ref. 41) being expressed in terms of the parameters c_i ($i = 1$, $2, \ldots, 13$) together with A, B, and C:

(i) The relevant constants:

$$c_4 = \frac{1}{2}(1 - c_1),\ c_5 = \frac{1}{2}(c_2 - 2c_3),\ c_6 = c_5^2 + A,$$

$$c_7 = 2c_4 c_5 - B,\ c_8 = c_4^2 + C,\ c_9 = c_3(c_7 + c_3 c_8) + c_6,$$

$$c_{10} = c_1 + 2c_4 + 2\sqrt{c_8} - 1 > -1, \; c_{11} = 1 - c_1 - 2c_4 + \frac{2}{c_3}\sqrt{c_9} > -1,$$

$$c_{12} = c_4 + \sqrt{c_8} > 0, \; c_{13} = -c_4 + \frac{1}{c_3}\left(\sqrt{c_9} - c_5\right) > 0. \tag{A1}$$

(ii) The essential polynomial functions:

$$\pi(z) = c_4 + c_5 z - \left[\left(\sqrt{c_9} + c_3\sqrt{c_8}\right)z - \sqrt{c_8}\right], \tag{A2}$$

$$k = -(c_7 + 2c_3 c_8) - 2\sqrt{c_8 c_9}, \tag{A3}$$

$$\tau(z) = 1 - (c_2 - 2c_5)z - 2\left[\left(\sqrt{c_9} + c_3\sqrt{c_8}\right)z - \sqrt{c_8}\right], \tag{A4}$$

$$\tau'(z) = -2c_3 - 2\left(\sqrt{c_9} + c_3\sqrt{c_8}\right) < 0. \tag{A5}$$

(iii) The energy equation:

$$(c_2 - c_3)n + c_3 n^2 - (2n+1)c_5 + (2n+1)\left(\sqrt{c_9} + c_3\sqrt{c_8}\right) + c_7 + 2c_3 c_8 + 2\sqrt{c_8 c_9} = 0. \tag{A6}$$

(iv) The wave functions:

$$\rho(z) = z^{c_{10}}(1 - c_3 z)^{c_{11}}, \tag{A7}$$

$$\phi(z) = z^{c_{12}}(1 - c_3 z)^{c_{13}}, \; c_{12} > 0, \; c_{13} > 0, \tag{A8}$$

$$y_n(z) = P_n^{(c_{10}, c_{11})}(1 - 2c_3 z), \; c_{10} > -1, \; c_{11} > -1, \tag{A9}$$

$$F_{n\kappa}(z) = \mathcal{N}_{n\kappa} z^{c_{12}}(1 - c_3 z)^{c_{13}} P_n^{(c_{10}, c_{11})}(1 - 2c_3 z), \tag{A10}$$

where $P_n^{(\mu,\nu)}(x)$, $\mu > -1$, $\nu > -1$, and $x \in [-1, 1]$ are the Jacobi polynomials with

$$P_n^{(\alpha,\beta)}(1 - 2s) = \frac{(\alpha+1)_n}{n!} \, _2F_1\left(-n, 1 + \alpha + \beta + n; \alpha + 1; s\right), \tag{A11}$$

and $\mathcal{N}_{n\kappa}$ is a normalization constants. Also, the above wave functions can be expressed in terms of the hypergeometric function as

$$F_{n\kappa}(z) = \mathcal{N}_{n\kappa} z^{c_{12}}(1 - c_3 z)^{c_{13}} \, _2F_1\left(-n, 1 + c_{10} + c_{11} + n; c_{10} + 1; c_3 z\right), \tag{A12}$$

where $c_{12} > 0$, $c_{13} > 0$ and $z \in [0, 1/c_3]$.

APPENDIX B: SOLVING QUARTIC ENERGY EQUATION

In order to solve the quartic equation (7b), the first step in the solution is to define the following variables:

$$u = a_2 - \frac{3}{8}a_3^2; \; v = a_1 + \frac{1}{8}a_3^3 - \frac{1}{2}a_3 a_2; \; w = a_0 - \frac{3}{256}a_3^4 + \frac{1}{16}a_3^2 a_2 - \frac{1}{4}a_3 a_1; \tag{B1}$$

which enable us to write down the related auxiliary cubic equation of the form:

$$a\overline{E}_{n\kappa}^3 + b\overline{E}_{n\kappa}^2 + c\overline{E}_{n\kappa} + d = 0, \tag{B2}$$

with coefficients

$$a = 1, \; b = \frac{u}{2}, \; c = \frac{1}{16}\left(u^2 - 4w\right), \; d = -\frac{v^2}{64}. \tag{B3}$$

The next step is solving the cubic equation (B2) by defining the variables

$$f = \frac{c}{a} - \frac{1}{3}\frac{b^2}{a^2}; \; g = \frac{2}{27}\frac{b^3}{a^3} - \frac{1}{3}\frac{bc}{a^2} + \frac{d}{a}; \; h = \frac{1}{4}g^2 + \frac{1}{27}f^3, \tag{B4}$$

and then applying one of the following three cases:

(i) When all three roots are real ($h < 0$): we define the variables

$$s = \sqrt{\frac{1}{4}g^2 - h};\; k = \cos^{-1}\left(-\frac{g}{2s}\right),\; F = \frac{1}{\sqrt[3]{s}},\; H = \cos\left(\frac{k}{3}\right),\; G = \sqrt{3}\sin\left(\frac{k}{3}\right), \quad \text{(B5)}$$

where all the arguments in the trigonometric functions are in radians, to obtain the three possible roots of (B2) in the form

$$\overline{E}_{n\kappa}^{(1)} = 2\frac{H}{F} - \frac{1}{3}\frac{b}{a};\; \overline{E}_{n\kappa}^{(2)} = F(H+G) + \frac{3a}{b};\; \overline{E}_{n\kappa}^{(3)} = F(H-G) + \frac{3a}{b}, \quad \text{(B6)}$$

and hence the four energies of the original quartic equation (7b) take the forms

$$E_{n\kappa}^{(1)} = \sqrt{\overline{E}_{n\kappa}^{(2)}} + \sqrt{\overline{E}_{n\kappa}^{(3)}} - \frac{g}{8\sqrt{\overline{E}_{n\kappa}^{(2)}\overline{E}_{n\kappa}^{(3)}}} - \frac{1}{4}\frac{a_3}{a_4};\; E_{n\kappa}^{(2)} = \sqrt{\overline{E}_{n\kappa}^{(2)}} - \sqrt{\overline{E}_{n\kappa}^{(3)}} + \frac{g}{8\sqrt{\overline{E}_{n\kappa}^{(2)}\overline{E}_{n\kappa}^{(3)}}} - \frac{1}{4}\frac{a_3}{a_4};$$

$$E_{n\kappa}^{(3)} = -\sqrt{\overline{E}_{n\kappa}^{(2)}} + \sqrt{\overline{E}_{n\kappa}^{(3)}} + \frac{g}{8\sqrt{\overline{E}_{n\kappa}^{(2)}\overline{E}_{n\kappa}^{(3)}}} - \frac{1}{4}\frac{a_3}{a_4};\; E_{n\kappa}^{(4)} = -\sqrt{\overline{E}_{n\kappa}^{(2)}} - \sqrt{\overline{E}_{n\kappa}^{(3)}} - \frac{g}{8\sqrt{\overline{E}_{n\kappa}^{(2)}\overline{E}_{n\kappa}^{(3)}}} - \frac{1}{4}\frac{a_3}{a_4}.$$
$$\text{(B7)}$$

It is worth noting that whenever we have three real root we always choose the two non-zero roots; say, $\overline{E}_{n\kappa}^{(2)}$ and $\overline{E}_{n\kappa}^{(3)}$ in (B6) of the cubic equation.

(ii) When only one root is real ($h > 0$): the definitions

$$R = -\frac{1}{2}g + \sqrt{h};\; S = \sqrt[3]{R};\; T = -\frac{1}{2}g - \sqrt{h};\; U = \sqrt[3]{T}; \quad \text{(B8)}$$

enable us to write down the three roots of (B2):

$$\overline{E}_{n\kappa}^{(1)} = S + U - \frac{b}{3a};\; \overline{E}_{n\kappa}^{(2)} = -\frac{1}{2}(S+U) - \frac{b}{3a} + i\frac{\sqrt{3}}{2}(S-U);$$

$$\overline{E}_{n\kappa}^{(3)} = -\frac{1}{2}(S+U) - \frac{b}{3a} - i\frac{\sqrt{3}}{2}(S-U), \quad \text{(B9)}$$

and hence the four energies of Eq. (7b) are

$$E_{n\kappa}^{(1)} = \sqrt{\overline{E}_{n\kappa}^{(2)}} + \sqrt{\overline{E}_{n\kappa}^{(3)}} - \frac{g}{8\sqrt{\overline{E}_{n\kappa}^{(2)}\overline{E}_{n\kappa}^{(3)}}} - \frac{1}{4}\frac{a_3}{a_4};\; E_{n\kappa}^{(2)} = \sqrt{\overline{E}_{n\kappa}^{(2)}} - \sqrt{\overline{E}_{n\kappa}^{(3)}} + \frac{g}{8\sqrt{\overline{E}_{n\kappa}^{(2)}\overline{E}_{n\kappa}^{(3)}}} - \frac{1}{4}\frac{a_3}{a_4};$$

$$E_{n\kappa}^{(3)} = -\sqrt{\overline{E}_{n\kappa}^{(2)}} + \sqrt{\overline{E}_{n\kappa}^{(3)}} + \frac{g}{8\sqrt{\overline{E}_{n\kappa}^{(2)}\overline{E}_{n\kappa}^{(3)}}} - \frac{1}{4}\frac{a_3}{a_4};\; E_{n\kappa}^{(4)} = -\sqrt{\overline{E}_{n\kappa}^{(2)}} - \sqrt{\overline{E}_{n\kappa}^{(3)}} - \frac{g}{8\sqrt{\overline{E}_{n\kappa}^{(2)}\overline{E}_{n\kappa}^{(3)}}} - \frac{1}{4}\frac{a_3}{a_4}.$$
$$\text{(B10)}$$

It is worth noting that whenever we have one real root and two complex roots we always choose the two complex roots.

(iii) When all three roots are real and equal ($f = g = h = 0$), then the roots of (B2):

$$\overline{E}_{n\kappa}^{(1)} = \overline{E}_{n\kappa}^{(2)} = \overline{E}_{n\kappa}^{(3)} = -\sqrt[3]{\frac{d}{a}}, \quad \text{(B11)}$$

and hence the four energies of the original quartic equation (7b) are

$$E_{n\kappa}^{(1)} = -2\sqrt[3]{\frac{d}{a}} - \frac{1}{4}\frac{a_3}{a_4};\; E_{n\kappa}^{(2)} = E_{n\kappa}^{(3)} = -\frac{1}{4}\frac{a_3}{a_4};\; E_{n\kappa}^{(4)} = 2\sqrt[3]{\frac{d}{a}} - \frac{1}{4}\frac{a_3}{a_4}. \quad \text{(B12)}$$

APPENDIX C: NORMALIZATION CONSTANTS

The normalization constant, $\mathcal{N}_{nl}$, can be determined in closed form. We start by using the relation between the hypergeometric function and the Jacobi polynomials (see formula (8.962.1) in Ref. 49):

$$_2F_1\left(-n, n+\nu+\mu+1; \nu+1; \frac{1-x}{2}\right) = \frac{n!}{(\nu+1)_n} P_n^{(\nu,\mu)}(x),$$

$$(\nu+1)_n = \frac{\Gamma(n+\nu+1)}{\Gamma(\nu+1)}, \tag{C1}$$

to rewrite the wave functions in (32) as

$$F_{n\kappa}(r) = \mathcal{N}_{n\kappa} \frac{n!\Gamma(2\varepsilon_{n\kappa}+1)}{\Gamma(n+2\varepsilon_{n\kappa}+1)} e^{-\varepsilon_{n\kappa}\alpha r}(1-e^{-\alpha r})^{\delta_1} P_n^{(2\varepsilon_{n\kappa},2\delta_1-1)}(1-2e^{-\alpha r}). \tag{C2}$$

From the normalization condition $\int_0^\infty \left[u_{n,l}(r)\right]^2 dr = 1$ and under the coordinate change $x = 1 - 2e^{-\alpha r}$, the normalization constant in (C2) is given by

$$\mathcal{N}_{n\kappa}^{-2} = \frac{1}{\alpha}\left[\frac{n!\Gamma(2\varepsilon_{n\kappa}+1)}{\Gamma(n+2\varepsilon_{n\kappa}+1)}\right]^2 \int_{-1}^1 \left(\frac{1-x}{2}\right)^{2\varepsilon_{n\kappa}} \left(\frac{1+x}{2}\right)^{2\delta_1-1} \left(\frac{1+x}{2}\right) \left[P_n^{(2\varepsilon_{n\kappa},2\delta_1-1)}(x)\right]^2 dx. \tag{C3}$$

The calculation of this integral can be done by writing

$$\frac{1+x}{2} = 1 - \left(\frac{1-x}{2}\right),$$

and using the following two integrals (see formula (7.391.5) in Ref. 49):

$$\int_{-1}^1 (1-x)^{\nu-1}(1+x)^\mu \left[P_n^{(\nu,\mu)}(x)\right]^2 dx = 2^{\nu+\mu}\frac{\Gamma(n+\nu+1)\Gamma(n+\mu+1)}{n!\nu\Gamma(n+\nu+\mu+1)}, \tag{C4}$$

which is valid for $Re(\nu) > 0$ and $Re(\mu) > -1$ and (see formula (7.391.1) in Ref. 49):

$$\int_{-1}^1 (1-x)^\nu(1+x)^\mu \left[P_n^{(\nu,\mu)}(x)\right]^2 dx = 2^{\nu+\mu+1}\frac{\Gamma(n+\nu+1)\Gamma(n+\mu+1)}{n!\Gamma(n+\nu+\mu+1)(2n+\nu+\mu+1)}, \tag{C5}$$

which is valid for $Re(\nu) > -1$, $Re(\mu) > -1$. Thus, the normalization constant:

$$\mathcal{N}_{n\kappa} = \frac{1}{\Gamma(2\varepsilon_{n\kappa}+1)}\left[\frac{\alpha\varepsilon_{n\kappa}(n+\varepsilon_{n\kappa}+\delta_1)}{2(n+\delta_1)}\frac{\Gamma(n+2\varepsilon_{n\kappa}+1)\Gamma(n+2\varepsilon_{n\kappa}+2\delta_1)}{n!\Gamma(n+2\delta_1)}\right]^{1/2}, \tag{C6}$$

where $0 \leq n, \kappa < \infty$. In the s-wave ($\kappa = -1$) case, the above result is written explicitly as

$$\mathcal{N}_{n,-1} = \frac{1}{\Gamma(2\eta+1)}\left[\frac{\alpha\eta(n+\eta+\delta)}{2(n+\delta)}\frac{\Gamma(n+2\eta+1)\Gamma(n+2\eta+2\delta)}{n!\Gamma(n+2\delta)}\right]^{1/2}. \tag{C7}$$

Also, the non-relativistic normalization constant is therefore obtained as

$$\mathcal{N}_{nl} = \frac{1}{\Gamma(2\widetilde{\eta}+1)}\left[\frac{\alpha\widetilde{\eta}(n+\widetilde{\eta}+\widetilde{\delta})}{2(n+\widetilde{\delta})}\frac{\Gamma(n+2\widetilde{\eta}+1)\Gamma(n+2\widetilde{\eta}+2\widetilde{\delta})}{n!\Gamma(n+2\widetilde{\delta})}\right]^{1/2}. \tag{C8}$$

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