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B_c SPECTROSCOPY IN THE SHIFTED l-EXPANSION TECHNIQUE

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In the framework of static and QCD-motivated model potentials for heavy quarkonium, we present a further comprehensive calculation of the mass spectrum of $\bar{b}c$ system and its ground state spin-dependent splittings in the context of the shifted l-expansion technique. We also predict the leptonic constant f_{B_c} of the lightest pseudoscalar B_c, and $f_{B_c^*}$ of the vector B_c^* states taking into account the one-loop and two-loop QCD corrections. Furthermore, we use the scaling relation to predict the leptonic constant of the nS-states of the $\bar{b}c$ system. Our predicted results are generally in high agreement with some earlier numerical methods. The parameters of each potential are adjusted to obtain best agreement with the experimental spin-averaged data (SAD).

Keywords: B_c-meson; mass spectrum; leptonic constant; hyperfine splittings and heavy quarkonium.

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1. Introduction

Recently, theoretical interest has risen in the study of the B_c-meson, the heavy bc quarkonium system with open charm and bottom quarks composing of two nonrelativistic heavy quarks.

The spectrum and properties of the bc systems have been calculated various times in the past in the framework of heavy quarkonium theory.[1] Moreover, the recent discovery[2] B_c meson (the lowest pseudoscalar $^1S_0$ state of the B_c system) opens up new theoretical interest in this subject.[1,3{12] The Collider Detector at Fermilab (CDF) Collaboration quotes M_{B_c} = $6.40^{\pm 0.39}_{\pm 0.13}$ GeV.[2]

This state should be one of a number of states lying below the threshold for emission of B and D mesons. Furthermore, such states are very stable in comparison with their counterparts in charmonium (cc) and upsilon (bb) systems. A particularly interesting quantity should be the hyper ne splitting that as for cc

case seems to be sensitive to relativistic and subleading corrections in the strong coupling constant α_s. For the above reasons it seems worthwhile to give a detailed account of the Schrödinger energies for cc, bb and bc meson systems below the continuum threshold. Because of the success of the nonrelativistic potential model and the flavor independence of the q_1q_2 potential, we choose a set of phenomenological and a QCD-motivated potentials by insisting upon strict flavor-independence of its parameters. We also use a potential model that includes running coupling constant effects in both the spherically symmetric potential and the spin-dependent potentials to give a simultaneous account of the properties of the cc, bb and bc meson systems. Since one would expect the average values of the momentum transfer in the various quark–antiquark states to be different, some variation in the values of the strong coupling constant and the normalization scale in the spin-dependent potential should be expected.[1,10–12]

This study is almost a full treatment for the potentials used in the literature. Therefore, in order to minimize the role of flavor-dependence, we use the same values for the coupling constant and the renormalization scale for each of the levels in a given system and require these values to be consistent with a universal QCD scale.

Kwong and Rosner[7] predicted the masses of the lowest vector and pseudoscalar states of the bc system using an empirical mass formula and a logarithmic potential. Eichten and Quigg[1] gave a more comprehensive account of the energies and decays of the B_c system that was based on the QCD-motivated potential of Buchmüller and Tye.[13] Gershtein *et al.*[8] also published a detailed account of the energies and decays of the B_c system using the QCD sum rule calculations. Baldicchi and Prosperi[6] have computed the bc spectrum based on an effective mass operator with full relativistic kinematics. They[6] have also fitted the entire light-heavy quarkonium spectrum. Fulcher[4] extended the treatment of the spin-dependent potentials to the full radiative one-loop level and thus included the effects of the running coupling constant in these potentials. He also used the renormalization scheme developed by Gupta and Radford.[14] Ebert *et al.*[1] comprehensively investigated the B_c meson masses and decays in the relativistic quark model. Very recently, we have reproduced the B_c meson spectroscopy and the bound-energy masses of mesons containing the fourth generation and iso-singlet quarks by employing the shifted large-N expansion technique (SLNET) using a group of static and improved QCD motivated potentials.[10,11]

One of the important objectives of the present work is to extend our previous works by using the shifted l-expansion technique (SLET)[15] developed for the Schrödinger equation to reproduce the bc spectroscopy[10,11] using a class of three static together with Martin and Logarithmic potentials[10–12] which have already been utilized[12] for the spin-averaged masses of the self-conjugate (QQ and qq) and also the non-self conjugate (Qq) mesons.[12] We also extend our work by using an improved QCD-motivated potential previously proposed by Buchmüller and Tye.[13]

The contents of this paper are as follows: in Sec. 2, we present the solution of the Schrödinger equation using the SLET for the non-self conjugate Qq meson mass spectrum. In Sec. 3, we brie y present all the potentials used in the present work. In Sec. 4, we present the rst-loop and second-loop correction of the B_c leptonic decay constant. Finally, our discussions and conclusions are given in Sec. 5.

2. The Method

In our previous papers[10–12] we have applied the shifted $1/N$ expansion technique (SLNET) to solve nonrelativistic and relativistic wave equations.[11,12] The method starts by writing the original wave equation in an N-dimensional space which is su ciently large and using the expansion $1/k$ as a perturbation parameter.[16] Here $k = N + 2l - a$, N is the number of spatial dimensions of interest, l is the angular quantum number, and a is a suitable shift as an additional degree of freedom and is responsible for speeding up the convergence of the resulting energy series. The main motivation of the present method is to overcome the shortcomings of the previous approaches and to formulate an elegant algebraic approach to yield a fairly simple analytic formula which gives rapidly converging leading-orders of the energy values with good accuracy. In this work another technique simply consists of using $1/l$ as an expansion parameter, where $l = l - a$, l is an angular quantum number and a is a suitable shift which is mainly introduced to avoid the trivial case $l = 0$. The choice of a is physically motivated so that the next to the leading energy eigenvalue series vanish as in SLNET. It suggests we should not worry about the N-dimensional form of the wave equation and we should expand directly through the quantum numbers involved in the problem. This method seems more exible and simple in treatment and has a quite di erent mathematical expansion than SLNET. Like SLNET, the shifted l-expansion technique (SLET) is also a pseudoperturbative technique. We feel encouraged to extend our previous works[10–12] using the SLET. We consider the radial part of the Schrödinger equation for an arbitrary spherically symmetric potential $V(r)$ (in units $\hbar = 1$)

$$\left\{-\frac{1}{4\mu}\frac{d^2}{dr^2} + \frac{l(l+1)}{4\mu r^2} + V(r)\right\}u(r) = E_{n,l}u(r)\,, \tag{1}$$

where $\mu = (m_q m_Q)/(m_q + m_Q)$ is the reduced mass for the two interacting particles and $E_{n,\ell}$ denotes the Schrödinger binding energy. Furthermore, Eq. (1) can be rewritten as

$$\left\{-\frac{1}{4\mu}\frac{d^2}{dr^2} + \frac{[l^2 + (2a+1)l + a(a+1)]}{4\mu r^2} + V(r)\right\}u(r) = E_{n,l}u(r)\,, \tag{2}$$

where $l = l - a$ with a representing a proper shift to be calculated later on and l is the angular quantum number. We follow the shifted l-expansion method[15] (expansion as $1/l$) by defining

$$V(y(r_0)) = \sum_{m=0}^{\infty}\left(\frac{d^m V(r_0)}{dr_0^m}\right)\frac{(r_0 y)^m}{m!Q} l^{-(m-4)/2}\,, \tag{3}$$

and also the energy eigenvalue expansion[10–12,15]

$$E_{n,l} = \sum_{m=0}^{\infty}\frac{l^{(2-m)}}{Q}E_m\,. \tag{4}$$

Here $y = l^{1/2}(r/r_0 - 1)$ with r_0 being an arbitrary point where the Taylor expansions is being performed about and Q is a scale to be set equal to l^2 at the end of our calculations. Inserting Eqs. (3) and (4) into Eq. (2) yields

$$\left\{-\frac{1}{4\mu}\frac{d^2}{dy^2} + \frac{1}{4\mu}\left[l + (2a+1) + \frac{a(a+1)}{l}\right]\sum_{m=0}^{\infty}\frac{(-1)^m(m+1)y^m}{l^{m/2}}\right.$$
$$\left. + \frac{r_0^2}{Q}\sum_{m=0}^{\infty}\left(\frac{d^m V(r_0)}{dr_0^m}\right)\frac{(r_0 y)^m}{m!}l^{(2-m)/2}\right\}\chi_{n_r}(y) = \xi_{n_r}\chi_{n_r}(y)\,. \tag{5}$$

Hence the final analytic expression for the $1/l$ expansion of the energy eigenvalues appropriate to the Schrödinger particle is[15]

$$\xi_{n_r} = \frac{r_0^2}{Q}\sum_{m=0}^{\infty} l^{(1-m)}E_m\,. \tag{6}$$

Now we formulate the SLET (expansion as $1/l$) for the nonrelativistic motion of spinless particle bound in spherically symmetric potential $V(r)$. On the other hand, the Schrödinger equation for a one-dimensional anharmonic-oscillator is[16]

$$\xi_{n_r} = l\left[\frac{1}{4\mu} + \frac{r_0^2 V(r_0)}{Q}\right] + \left[\left(n_r + \frac{1}{2}\right)\omega + \frac{(2a+1)}{4\mu}\right]$$
$$+ \frac{1}{l}\left[\frac{a(a+1)}{4\mu} + \gamma^{(1)}\right] + \frac{\gamma^{(2)}}{l^2} + O\left(\frac{1}{l^3}\right)\,, \tag{7}$$

where $\gamma^{(1)}$ and $\gamma^{(2)}$ are two expressions given explicitly in Appendix A. Thus, comparing Eq. (6) with Eq. (7) gives

$$E_0 = V(r_0) + \frac{Q}{4\mu r_0^2}\,, \tag{8}$$

$$E_1 = \frac{Q}{r_0^2}\left[\left(n_r + \frac{1}{2}\right)\omega + \frac{(2a+1)}{4\mu}\right]\,, \tag{9}$$

$$E_2 = \frac{Q}{r_0^2}\left[\frac{a(a+1)}{4\mu} + \gamma^{(1)}\right]\,, \tag{10}$$

and

$$E_3 = \frac{Q}{r_0^2}\gamma^{(2)} . \tag{11}$$

The quantity r_0 is chosen as to minimize the leading term, E_0, that is,[12]

$$\frac{dE_0}{dr_0} = 0 \qquad \text{and} \qquad \frac{d^2E_0}{dr_0^2} > 0 , \tag{12}$$

which yields the relation

$$Q = 2\mu r_0^3 V'(r_0) . \tag{13}$$

Further, to solve the shifting parameter a, the next contribution to the energy eigenvalues is chosen to vanish,[10–12,15,16] i.e., $E_1 = 0$, which provides smaller contributions for the higher-order corrections in (4) compared to the leading term contribution (8). It implies that the energy states are being calculated by considering only the leading term E_0, the second-order E_2 and the third-order E_3 corrections. So, the shifting parameter is determined via

$$a = -\frac{[1 + 2\mu(2n_r + 1)\omega]}{2} , \tag{14}$$

with

$$\omega = \frac{1}{2\mu}\left[3 + \frac{r_0 V''(r_0)}{V'(r_0)}\right]^{1/2} . \tag{15}$$

Therefore, the Schrodinger binding energy (4) to the third order is

$$E_{n,l} = V(r_0) + \frac{1}{r_0^2}\left[\frac{a(a+1) + Q}{4\mu} + \gamma^{(1)} + \frac{\gamma^{(2)}}{l} + O\left(\frac{1}{l^2}\right)\right] . \tag{16}$$

Furthermore, setting $l = \sqrt{Q}$ rescales the potential, we derive an analytic expression that is satisfying r_0 as

$$2l + \left\{1 + (2n_r + 1)\left[3 + \frac{r_0 V''(r_0)}{V'(r_0)}\right]^{1/2}\right\} = 2\left[2\mu r_0^3 V'(r_0)\right]^{1/2} , \tag{17}$$

where $n_r = n - 1$ is the radial quantum number. Once r_0 is being found through Eq. (17) for any arbitrary state, the determination of the binding energy for the Qq system becomes relatively simple and straightforward. Finally, the Schrodinger binding mass can be determined by

$$M(Qq) = m_q + m_Q + 2E_{n,l} . \tag{18}$$

It is being found that for a fixed n, the computed energies become more accurate as l increases.[10–12,15,16] This is expected since the expansion parameter $1/l$ becomes smaller as l becomes larger since the parameter l is proportional to n and appears in the denominator in higher-order correction.

3. Some Model Potentials

The bc system that we investigate is often considered as nonrelativistic system and consequently our treatment is based upon Schrödinger equation with a Hamiltonian

$$H_0 = -\frac{\nabla^2}{4\mu} + V(\mathrm{r}) + V_{\mathrm{SD}}\,, \tag{19}$$

where we have supplemented our nonrelativistic Hamiltonian with the standard spin-dependent terms[1,10,11,17]

$$V_{\mathrm{SD}} \to V_{\mathrm{SS}} = \frac{32\pi\alpha_s}{9m_q m_Q}(\mathrm{s}_1 \cdot \mathrm{s}_2)\delta^3(\mathrm{r})\,. \tag{20}$$

Here, the spin dependent potential is simply a spin-spin part[1,17] that would enable us to make some preliminary calculations of the energies of the lowest two S-states of the bc system. The potential parameters in this section are all strictly flavor-independent and fitted to the low-lying energy levels of cc and bb systems. Like most authors (cf. e.g. Ref. 1), we determine the coupling constant $\alpha_s(m_c^2)$ from the well measured hyperfine splitting of the $1S(cc)$ state[17]

$$E_{\mathrm{hfs}} = M_{J/\psi} - M_{\eta_c} = 117 \pm 2\ \mathrm{MeV}\,, \tag{21}$$

for each desired potential to produce the center-of-gravity (cog) of the $M_\psi(1S)$ value. The numerical value of α_s is found to be dependent on the potential form and also be compatible with the other measurements.[1,3,4,6–8] Therefore, the $1S$-state hyperfine splitting[10,11,17] is given by[a]

$$E_{\mathrm{hfs}} = \frac{8\alpha_s(\mu)}{9m_c m_b}|R_{1S}(0)|^2\,, \tag{22}$$

with the radial wave function originally determined via[10,11,17]

$$|R_{1S}(0)|^2 = 2\mu\left\langle \frac{dV(r)}{dr} \right\rangle\,. \tag{23}$$

Hence, the total mass of the low-lying pseudoscalar B_c meson is[10]

$$M_{B_c}(0^-) = m_c + m_b + 2E_{1,0} - 3\ E_{\mathrm{hfs}}/4\,, \tag{24}$$

and for the vector B_c^* meson

$$M_{B_c^*}(1^-) = m_c + m_b + 2E_{1,0} + \ E_{\mathrm{hfs}}/4. \tag{25}$$

Hence, the square-mass difference can be simply found as

$$M^2 = M^2_{B_c^*}(1^-) - M^2_{B_c}(0^-) = 2\ E_{\mathrm{HF}}(m_c + m_b + 2E_{1,0} - \ E_{\mathrm{HF}}/4)\,. \tag{26}$$

[a]At present, the only measured splitting of nS-levels is that of η_c and J/ψ, which allows us to evaluate the so-called SAD using $\bar{M}_\psi(1S) = (3M_{J/\psi} + M\eta_c)/4$ and also $\bar{M}(nS) = M_V(nS) - (M_{J/\psi} - M\eta_c)/4n$.[17,18]

The perturbative part of such a quantity was evaluated at the lowest order in α_s. Baldicchi and Prosperi[6] used the standard running QCD coupling expression

$$\alpha_s(\mathrm{Q}) = \frac{4\pi}{\left(11 - \frac{2}{3}n_f\right)\ln\left(\frac{\mathrm{Q}^2}{\Lambda^2}\right)}, \tag{27}$$

with $n_f = 4$ and = 0.2 GeV cut at a maximum value $\alpha_s(0) = 0.35$, to give the right $J/\psi - \eta_c$ splitting and to treat properly the infrared region.[6] Furthermore, Brambilla and Vairo[3] took in their perturbative analysis $0.26 \le \alpha_s(\mu = 2\ \text{GeV}) \le 0.30$. Badalian *et al.*[19] used $\alpha_s(\mu = 0.92\ \text{GeV}) \simeq 0.36$ for all states, but the splittings do not practically change if $\alpha_s(\mu = 1.48\ \text{GeV}) = 0.30$ is taken. Furthermore, Motyka and Zalewski[20] found $\alpha_s(m_c^2) = 0.3376$ and from which they calculated $\alpha_s(m_b^2) = 0.2064$ and $\alpha_s(4\mu_{bc}^2) = 0.2742$.

3.1. *Static potentials*

The potential in Eq. (19) includes a class of static potentials previously proposed by Lichtenberg[21]

$$V(r) = -ar^{-\beta} + br^{\beta} + c; \qquad 0 < \beta \le 1, \tag{28}$$

where $a > 0$, $b > 0$ and c may be of either sign. These static quarkonium potentials are monotone nondecreasing and concave functions which satisfy the condition

$$V'(r) > 0 \ \text{ and } \ V''(r) \le 0. \tag{29}$$

This comprises a wide class of potentials presented in our previous works.[10{12]

3.1.1. *Cornell potential*

The QCD-motivated Coulomb-plus-linear potential (Cornell potential)[22]

$$V_C(r) = -\frac{a}{r} + br + c, \tag{30}$$

with the adjustable set of parameters

$$[a, b, c] = [0.52, 0.1756\ \text{GeV}^2, \ \ -0.8578\ \text{GeV}]. \tag{31}$$

3.1.2. *Song–Lin potential*

This phenomenological potential was proposed by Song and Lin[23] with the form

$$V_{\mathrm{SL}}(r) = -ar^{-1/2} + br^{1/2} + c, \tag{32}$$

with the adjustable set of parameters

$$[a, b, c] = [0.923\ \text{GeV}^{1/2}, \quad 0.511\ \text{GeV}^{3/2}, \quad -0.798\ \text{GeV}]. \tag{33}$$

3.1.3. *Turin potential*

Lichtenberg *et al.*[21] suggested that such a potential is an intermediate between the Cornell and Song{Lin potentials with the form

$$V_T(r) = -ar^{-3/4} + br^{3/4} + c\,, \tag{34}$$

with the adjustable values of parameters

$$[a, b, c] = [0.620\ \text{GeV}^{1/4}, \quad 0.304\ \text{GeV}^{7/4}, \quad -0.823\ \text{GeV}]\,. \tag{35}$$

3.1.4. *Martin potential*

The phenomenological power-law potential of the form[24,25]

$$V_M(r) = b(\quad_M r)^{0.1} + c\,, \tag{36}$$

is labeled as Martin's potential[24] with the values of parameters (potential units are also in GeV)

$$[b, c, \quad_M] = [6.898\ \text{GeV}^{1.1}, \quad -8.093\ \text{GeV}, \quad 1\ \text{GeV}]\,. \tag{37}$$

3.1.5. *Logarithmic potential*

A Martin's power-law potential reduces into the form[24]

$$V_L(r) = b\ln(\quad_L r) + c\,, \tag{38}$$

with

$$[b, c, \quad_L] = [0.733\ \text{GeV}, \quad -0.6631\ \text{GeV}, \quad 1\ \text{GeV}]\,. \tag{39}$$

The potential forms in (36) and (38) were used by Eichten *et al.*[1] and Kiselev.[25] Further, all of these potential forms were also used for ψ and   data probing 0.1 fm $< r <$ 1 fm region.[25] The characteristic feature of these potentials may be traced in Refs. 10 and 11.

3.2. *QCD-motivated potentials*

3.2.1. *Igi–Ono potential*

Igi and Ono[13,26] proposed a potential which is consisting of two parts, the short distance interquark one-gloun exchange part of the form

$$V_{\text{OGE}}^{(n_f=4)}(r) = -\frac{16\pi}{25}\frac{1}{rf(r)}\left[1 - \frac{462}{625}\frac{\ln f(r)}{f(r)} + \frac{2\gamma_E + \frac{53}{75}}{f(r)}\right], \tag{40}$$

with

$$f(r) = \ln\left[\frac{1}{r^2 \quad ^2_{\bar{M}S}} + b\right], \tag{41}$$

where n_f is the number of avors with mass below μ, and $\gamma_E = 0.5772$ is the Euler's number. Moreover, the long distance interquark potential grows linearly leading to con nement as

$$V_L(r) = ar\,. \tag{42}$$

Therefore, the Igi{Ono potential is[26]

$$V^{(n_f=4)}(r) = V_{\rm OGE}^{(n_f=4)} + ar + dre^{-gr}\,, \tag{43}$$

where the term dre^{-gr} in (43) is added to interpolate smoothly between the two parts and to adjust the intermediate range behavior by which the range of $_{\bar{M}S}$ is extended to keep linearly rising con ning potential. Numerical calculations show that potential is good for $_{\bar{M}S}$ in the range 100{500 MeV to keep a good t to the cc and bb spectra. Thereby, the potential with $b = 20$ is labeled as type I, the one with $b = 5$ is labeled as type II. Their adjusted parameters are given in Table 3 of our previous work.[11] Furthermore, the potential with $a = 0.1414$ GeV, $d = g = 0$, and $b = 19$ is labeled as type III.[11]

3.2.2. *Improved Chen–Kuang potential*

Chen and Kuang[27] proposed two improved potential models so that the parameters therein all vary explicitly with $_{\bar{M}S}$, therefore these parameters can only be given numerically for several values of $_{\bar{M}S}$. Such potentials have the natural QCD interpretation and explicit $_{\bar{M}S}$ dependence both for giving clear link between QCD and experiments and for convenience in practical calculation for a given value of $_{\bar{M}S}$. It has the general form

$$V^{(n_f=4)}(r) = kr - \frac{16\pi}{25}\frac{1}{rf(r)}\left[1 - \frac{462}{625}\frac{\ln f(r)}{f(r)} + \frac{2\gamma_E + \frac{53}{75}}{f(r)}\right]\,, \tag{44}$$

where the string tension is related to Regge slope by $k = \frac{1}{2\pi\alpha}$. The function $f(r)$ in (44) can be read o from

$$f(r) = \ln\left[\frac{1}{_{\bar{M}S}r} + 5.10 - A(r)\right]^2\,, \tag{45}$$

with

$$A(r) = \left[1 - \frac{1}{4}\frac{_{\bar{M}S}}{^{I}_{\bar{M}S}}\right]\frac{1 - \exp\left\{-\left[15\left(3\frac{\Lambda^I_{\bar{M}S}}{\Lambda_{\bar{M}S}} - 1\right)\ _{\bar{M}S}r\right]^2\right\}}{_{\bar{M}S}r}\,. \tag{46}$$

The scale parameter $^{I}_{\bar{M}S}$ is very close to the value of $_{\bar{M}S}$ determined from the two-photon processes and is also close to the world-averaged value of $_{\bar{M}S}$. The tted values of its parameters are as follows

$$\left[k, \alpha',\ \ \frac{I}{MS}\right] = [0.1491\ {\rm GeV}^2,\quad 1.067\ {\rm GeV}^{-2},\quad 180\ {\rm MeV}]\,. \tag{47}$$

The details of this potential can be traced in Ref. 27 and the tted quark masses are also displayed in Ref. 11.

4. Leptonic Constant of the B_c-Meson

The study of the heavy quarkonium system has played a vital role in the development of the QCD. Some of the earliest applications of perturbative QCD were calculations of the decay rates of charmonium.[28] These calculations were based on the assumption that, in the nonrelativistic (NR) limit, the decay rate factors into a short-distance (SD) perturbative part associated with the annihilation of the heavy quark and antiquark, and a long-distance (LD) part associated with the quarkonium wavefunction. Calculations of the annihilation decay rates of heavy quarkonium have recently been placed on a solid theoretical foundation by Bodwin *et al.*[29] Using NRQCD[30] to seperate the SD and LD e ects, Bodwin *et al.* derived a general factorization formula for the inclusive annihilation decay rates of heavy quarkonium. The SD factors in the factorization formula can be calculated using pQCD,[18] and the LD factors are de ned rigorously in terms of the matrix elements of NRQCD that can be estimated using lattice calculations.[5] It applies equally well to S-wave, P-wave, and higher orbital-angular-momentum states, and it can be used to incorporate relativistic corrections to the decay rates.

In the NRQCD[30] approximation for the heavy quarks, the calculation of the leptonic decay constant for the heavy quarkonium with the two-loop accuracy requires the matching of NRQCD currents with corresponding full-QCD axial-vector currents[32]

$$\mathcal{J}^{\lambda}|_{\mathrm{NRQCD}} = -\chi_b^{\dagger}\psi_c v^{\lambda} \quad \text{and} \quad J^{\lambda}|_{\mathrm{QCD}} = b\gamma^{\lambda}\gamma_5 c\,, \tag{48}$$

where b and c are the relativistic bottom and charm elds, respectively, $\chi_b^{\dagger}$ and ψ_c are the NR spinors of anti-bottom and charm, and v^{λ} is the four-velocity of heavy quarkonium. The NRQCD lagrangian describing the B_c-meson bound state dynamics is[33]

$$\mathcal{L}_{\mathrm{NRQCD}} = \mathcal{L}_{\mathrm{light}} + \psi_c^{\dagger}(iD_0 + \mathbf{D}^2/(2m_c))\psi_c + \chi_b^{\dagger}(iD_0 - \mathbf{D}^2/(2m_b))\chi_b + \cdots\,, \tag{49}$$

where $\mathcal{L}_{\mathrm{light}}$ is the relativistic lagrangian for gluons and light quarks. The two-component spinor eld ψ_c annihilates charm quarks, while χ_b creates bottom antiquarks. The relative velocity v of heavy quarks inside the B_c-meson provides a small parameter that can be used as a nonperturbative expansion parameter. To express the decay constant f_{B_c} in terms of NRQCD matix elements we express $J^{\lambda}|_{\mathrm{QCD}}$ in terms of NRQCD elds ψ_c and χ_b. Only the $\lambda = 0$ current-component contributes to the matrix element in the rest frame of the B_c-meson:

$$\langle 0|b\gamma^{\lambda}\gamma_5 c|B_c(\mathbf{P})\rangle = if_{B_c}P^{\lambda}\,, \tag{50}$$

where $|B_c(\mathbf{P})\rangle$ is the state of the with four-momentum P. It has the standard covariant normalization

$$\frac{1}{(2\pi)^3}\int \psi_{B_c}^{*}(p')\psi_{B_c}(p)d^3p = 2E\delta^{(3)}(p'-p)\,, \tag{51}$$

and its phase has been chosen so that f_{B_c} is real and positive. Hence, the matching yields

$$b\gamma^0\gamma_5 c = K_0\chi_b^\dagger\psi_c + K_2(\mathbf{D}\chi_b)^\dagger \cdot \mathbf{D}\psi_c + \cdots, \tag{52}$$

where $K_0 = K_0(m_c, m_b)$ and $K_2 = K_2(m_c, m_b)$ are Wilson SD coe cients. They can be determined by matching perturbative calculations of the matrix element $\langle 0|b\gamma^0\gamma_5 c|B_c\rangle$, a contribution is mostly coming up from the rst term in

$$\begin{aligned}\langle 0|b\gamma^0\gamma_5 c|B_c\rangle|_{\mathrm{QCD}} &= K_0\langle 0|\chi_b^\dagger\psi_c|B_c\rangle|_{\mathrm{NRQCD}} \\ &\quad + K_2\langle 0|(\mathbf{D}\chi_b)^\dagger\cdot\mathbf{D}\psi_c|B_c\rangle|_{\mathrm{NRQCD}} + \cdots,\end{aligned} \tag{53}$$

where the matrix element on the left side of (53) is taken between the vacuum and the state $|B_c\rangle$. Hence, equation (53) can be estimated as:

$$|\langle 0|\chi_b^\dagger\psi_c|B_c\rangle|^2 \simeq \frac{3M_{B_c}}{\pi}|R_{1S}(0)|^2. \tag{54}$$

Onishchenko and Veretin[33] calculated the matrix elements on both sides of Eq. (53) up to α_s^2 order. Therefore, in one-loop calculation, they found the SD-coe cients:

$$K_0 = 1 \text{ and } K_2 = -\frac{1}{8\mu^2}, \tag{55}$$

with μ being de ned after Eq. (1). Furthermore, Braaten and Fleming (BF) in their work[34] calculated the perturbation correction to K_0 up to order α_s (one-loop correction) as

$$K_0 = 1 + c_1\frac{\alpha_s(\mu)}{\pi}, \tag{56}$$

with c_1 being calculated in Ref. 34 as

$$c_1 = -\left[2 - \frac{m_b - m_c}{m_b + m_c}\ln\frac{m_b}{m_c}\right]. \tag{57}$$

Finally, the leptonic decay constant for the one-loop calculations is

$$f_{B_c}^{(1\text{-loop})} = \left[1 - \frac{\alpha_s(\mu)}{\pi}\left(2 - \frac{m_b - m_c}{m_b + m_c}\ln\frac{m_b}{m_c}\right)\right] f_{B_c}^{\mathrm{NR}}, \tag{58}$$

where the NR leptonic constant[35] is given by

$$f_{B_c}^{\mathrm{NR}} = \sqrt{\frac{3}{\pi M_{B_c}}}|R_{1S}(0)| \tag{59}$$

and μ is any scale of order m_b or m_c of the running coupling constant. On the other hand, the calculations of two-loop correction in the case of vector current and equal quark masses were done in Ref. 36. Furthermore, Onishchenko and Veretin[33] extended the work of Ref. 36 into the non-equal mass case. They found an expression for the two-loop QCD corrections to B_c-meson leptonic constant which is given by

$$K_0(\alpha_s, M/\mu) = 1 + c_1(M/\mu)\frac{\alpha_s(M)}{\pi} + c_2(M/\mu)\left(\frac{\alpha_s(M)}{\pi}\right)^2 + \cdots, \tag{60}$$

where $c_1(M/\mu)$ is explicitly given in Eq. (57) and the two-loop matching coe cient, $c_2(M/\mu)$, is given in Ref. 33; Eqs. (16){(20). In the case of B_c-meson and pole quark masses (m_b = 4.8 GeV, m_c = 1.65 GeV), they found

$$f_{B_c}^{(2\text{-loop})} = \left[1 - 1.48\left(\frac{\alpha_s(m_b)}{\pi}\right) - 24.24\left(\frac{\alpha_s(m_b)}{\pi}\right)^2\right] f_{B_c}^{\rm NR}\,. \tag{61}$$

Here, the two-loop correction is large and constitutes nearly 100% of one-loop correction as stated in Ref. 33.

5. Results and Conclusions

We solve the Schrödinger equation for di erent phenomenological and QCD-motivated potentials. With the help of Eq. (21), we determine the position of the charmonium center-of-gravity $M_\psi(1S)$ mass spectrum. Furthermore, we x the coupling constant $\alpha_s(m_c)$ for each potential. For simplicity we neglect the variation of α_s with momentum in (27) to have a common spectra for all states and scale the splitting of bc and bb from the charmonium value in (21). The consideration of the variation of the e ective Coulomb interaction constant becomes especially essential for the particle, for which $\alpha_s(\ ) \neq \alpha_s(\psi)$.[b] So, we follow our previous works[10,11] to calculate the corresponding low-lying center-of-gravity $M_\Upsilon(1S)$ and consequently the low-lying $M_{B_c}(1S)$. Thus, in calculating the splittings of the bc spectra, we have to take into account the $\alpha_s(\mu)$ dependence on the reduced mass of the heavy quarkonium instead of $\alpha_s(\mathrm{Q})$ for the reasons stated in Ref. 3. That is, the QCD coupling constant α_s in (27) is de ned in the Gupta-Radford (GR) renormalization scheme[14]

$$\alpha_s = \frac{6\pi}{(33 - 2n_f)\ln\left(\frac{\mu}{\Lambda_{\rm GR}}\right)}\,, \tag{62}$$

in which $_{\rm GR}$ is related to $_{\bar{M}S}$ by

$$_{\rm GR} = {}_{\bar{M}S}\exp\left[\frac{49 - 10n_f/3}{2(33 - 2n_f)}\right]\,. \tag{63}$$

Taking the momentum dependence of Baldicchi *et al.* [cf. Eq. (27)] into account would increase the accuracy and probably reproduce the experimental values equally well within the errors.

Table 1 reports our prediction for the Schrödinger mass spectrum of the four lowest cb S-states together with the rst three P- and D-states below their strong decay threshold for di erent static potentials. Since the model is spin independent and as the energies of the singlet states of quarkonium families have not been

[b]Kiselev *et al.*[25] have taken into account that $\Delta M_\Upsilon(1S) = \frac{\alpha_s(\Upsilon)}{\alpha_s(\psi)}\Delta M_\psi(1S)$ with $\alpha_s(\Upsilon)/\alpha_s(\psi) \simeq 3/4$. Furthermore, Motyka and Zalewski[20] also found $\frac{\alpha_s(m_b^2)}{\alpha_s(m_c^2)} \simeq 11/18$.

Table 1. The $\bar{b}c$ masses and hyperfine splittings (Δ_{nS}) calculated in different static potentials (in MeV).

States	Refs. 1, 6	Cornell	Song–Lin	Turin	Martin	Logarithmic
$\alpha_s(m_c^2)$		0.320	0.263	0.286	0.251	0.220
m_c(GeV)		1.840	1.820	1.790	1.800	1.500
m_b(GeV)		5.232	5.199	5.171	5.174	4.905
$M(\bar{b}c)$						
$1S$	6315	6315	6306	6307	6301	6317
1^3S_1	6334	6335	6325	6326	6319	6334
1^1S_0	6258	6252	6249	6249	6247	6266
Δ_{1S}[a]	77	83.5	76.1	76.7	71.6	68.0
$2S$	6873	6888	6875	6880	6892	6903
2^3S_1	6883	6897	6884	6889	6902	6911
2^1S_0	6841	6860	6850	6852	6865	6879
Δ_{2S}	42	37.9	34.0	36.5	36.7	31.3
$3S$	7246	7271	7209	7246	7236	7225
$4S$		7587	7455	7535	7483	7448
$1P$	6772	6743	6733	6731	6730	6754
$2P$	7154	7138	7104	7123	7125	7127
$3P$		7464	7371	7428	7398	7375
$1D$	7043	7003	6998	6998	7011	7027
$2D$	7367	7340	7284	7320	7311	7301
$3D$		7636	7510	7588	7536	7502

[a] $\Delta_n S = M(n^3S_1) - M(n^1S_0)$.

measured,[11,18,21] a theoretical estimates of these unknown levels introduce uncertainty into the calculated SAD.[c] Our results in Table 1 for the B_c and B_c^* meson masses are in a pretty good agreement with the other authors.[1,4,7,11] Here, we report the range of the strong coupling constant at the m_c scale we take in our analysis $0.1985 \leq \alpha_s(m_c^2) \leq 0.320$ for all types of potentials and $0.220 \leq \alpha_s(m_c^2) \leq 0.320$ for the class of static potentials. In this model, we point out a di erent choice of the potential which would in general lead to a di erent value of the wave function at the origin and a di erent determination of $\alpha_s(m_c^2)$ from the same hyper ne splitting. Furthermore, our predictions to the bc masses of the lowest S-wave (singlet and triplet) together with the other estimations by many authors are given in Table 2. Larger discrepancies among the various methods occur for the ground and excited states.[6] Furthermore, Table 2 reports the binding masses of the singlet and triplet states and also the hyper ne splitting of the ground state together with those of other authors. Moreover, in Table 3, we also estimate the radial wave function of

[c]It is worthwhile to note that SAD is defined as the average mass of the ($s = 1$, $l = 1$) states in the form $\mathrm{SAD}(nP_j) = \frac{1}{9}[5M(n^3P_2) + 3M(n^3P_1) + M(n^3P_0)]$ and for ($s = 1$, $l = 0$) states by $\mathrm{SAD}(nS_j = \frac{1}{4}[3M(\mathrm{n}^3S_1) + M(\mathrm{n}^1S_0)]$, in which the SAD S-level gives the weight of only 1/4 to the unknown singlet level and 3/4 to the known triplet level.[11]

the low-lying state of the bc system, so that we have

$$|R_{1S}(0)| = 1.280 - 1.540 \text{ GeV}^{3/2}, \tag{64}$$

for the group of static potentials. Furthermore, we present our results for the NR leptonic constant $f_{B_c}^{\mathrm{NR}} = 466^{+19}_{-25}$ MeV and $f_{B_c^*}^{\mathrm{NR}} = 463^{+19}_{-24}$ MeV as an estimation of the potential models without the matching.[4,18] Our results are compared with those of Gershtein *et al.*,[37] who used Martin's potential, those of Ebert *et al.*,[1] and also with those of Jones and Woloshyn (JW).[38] Moreover, the one-loop correction, $f_{B_c}^{(1\text{-loop})}$ and the two-loop correction, $f_{B_c}^{(2\text{-loop})}$ are also given in Table 3. Hence, in the view of our results, the prediction for the one-loop calculations is

$$f_{B_c}^{(1\text{-loop})} = 408^{+16}_{-14} \text{ MeV} \quad \text{and} \quad f_{B_c^*}^{(1\text{-loop})} = 405^{+17}_{-14} \text{ MeV}, \tag{65}$$

and for two-loop calculations

$$f_{B_c}^{(2\text{-loop})} = 315^{+16}_{-51} \text{ MeV} \quad \text{and} \quad f_{B_c^*}^{(2\text{-loop})} = 313^{+26}_{-51} \text{ MeV}. \tag{66}$$

So, our numerical value for $f_{B_c}^{\mathrm{NR}}$ is in agreement with the estimates obtained in the framework of the lattice QCD result,[5] $f_{B_c}^{\mathrm{NR}} = 440 \pm 20$ MeV, QCD sum rules,[39] potential models,[1,4,18] and the scaling relation.[25] It indicates that the one-loop matching[32] provides the magnitude of correction of nearly 12%. Further, the most recent calculation[32] in the heavy quark potential in the static limit of QCD with the one-loop matching is

$$f_{B_c}^{(1\text{-loop})} = 400 \pm 15 \text{ MeV}. \tag{67}$$

Table 2. The predicted $\bar{b}c$ masses of the lowest S-wave and its splitting compared with the other authors (in MeV).

Work[a]	$M_{B_c}(1^1S_0)$[b]	$M_{B_c^*}(1^3S_1)$	Δ_{1S}
Eichten *et al.*[1]	6258 ± 20		
Colangelo and Fazio[3]	6280	6350	
Chabab[43]	6250 ± 200		
Baker *et al.*[44]	6287	6372	
Roncaglia *et al.*[45]		6320 ± 10	
Godfrey *et al.*[9]	6270	6340	
Bagan *et al.*[1,46]	6255 ± 20	6330 ± 20	
Brambilla *et al.*[3]		6326^{+29}_{-9}	60[c]
Baldicchi *et al.*[6]	$6194 \sim 6292$	$6284 \sim 6357$	$65 \le \Delta_{1S} \le 90$
SLET[d]	6253^{+13}_{-6}	6328^{+7}_{-9}	$68 \le \Delta_{1S} \le 83$
SLET[e]	6258^{+8}_{-11}	6333^{+2}_{-14}	

[a]The prediction is done by using two versions of QCD sum rules.
[b]The experimental mass of the singlet state is given in Ref. 2.
[c]Here we cite Ref. 5.
[d]Averaging over the five values in Table 1.
[e]We treat Eichten and Quigg's results in the same manner, see Ref. 1.

Table 3. The characteristics of the radial wave function at the origin $|R_{1S}(0)|^2$ (in GeV^3), NR, one-loop and two-loop corrections to pseudoscalar and vector decay constants of the low-lying B_c-meson (the accuracy is 5%) alculated in different static potential models (in MeV).

Quantity	Cornell	Song–Lin	Turin	Martin	Logarithmic	GKLT[37]	EFG[1]	JW[38]
$\lvert\psi_{1S}(0)\rvert^2$	0.112	0.123	0.111	0.119	0.102			
$\lvert R_{1S}(0)\rvert^2$	1.413	1.54	1.397	1.495	1.28			
$f_{B_c}^{(\mathrm{NR})}$	464.5	485.1	462.0	478.0	441.7	460 ± 60	433	420 ± 13
$f_{B_c^*}^{(\mathrm{NR})}$	461.5	482.2	459.2	475.3	439.3	460 ± 60	503	
$f_{B_c}^{(1\text{-loop})}$	393.6[a]	424.4	399.6	421.2	399.3			
$f_{B_c}^{(2\text{-loop})}$	264.1[b]	333.0	296.6	339.1	340.9			
$f_{B_c^*}^{(1\text{-loop})}$	391.0	421.9	397.1	418.8	397.2			
$f_{B_c^*}^{(2\text{-loop})}$	262.3	331.0	294.8	337.2	339.0			

[a]First loop SD Wilson coefficient for all potentials, $K_0 = 0.85 - 0.90$.
[b]Second loop SD Wilson coefficient for all potentials, $K_0 = 0.57 - 0.77$.

Therefore, in contrast to the discussion given in Ref. 32, we see that the di erence is not crucially large in our estimation to one-loop value in the B_c meson. On the other hand, our nal result of the two-loop calculations is

$$f_{B_c}^{(2\text{-loop})} = 315^{+26}_{-50}\ \text{MeV}\,, \tag{68}$$

the larger error value in (68) is due to the strongest running coupling constant in Cornell potential. Moreover, Motyka and Zalewski[20] also found $f_{B_c}^{(1\text{-loop})} = 435$ MeV for the ground state of bc quarkonium.

In the potential model, we note that slightly di erent additive constants are permitted to bring up data to their center-of-gravity values. However, with no additive constant to the Cornell potential,[40] we notice that the smaller mass value for the composing quarks of the meson leads to a rise in the values of the potential parameters which in turn produces a notable lower value for the leptonic constant.

Our predictions for the bc mass spectrum for the Igi{Ono potential (type I and II) are given in Table 4. Moreover, the singlet and triplet masses together with the hyper ne splittings predicted for the two types of this potential are also reported in Table 5. We, hereby, tested acceptable parameters for $_{\bar{M}S}$ from 100 to 500 MeV for the type I and II potentials[d] to produce the bc masses and their splittings. Small discrepancies between our prediction and SAD experiments[11,18,21,41] can be seen for higher states and such discrepancies are probably seen for any potential model and it might be related to the threshold e ects or quark-gloun mixings. The tted set of parameters for the Igi{Ono potential (type III)[11] are also tested in our

[d]The parameters of this potential are given in Table 3 of Ref. 11.

Table 4. The $\bar{b}c$ mass spectra predicted for various $\Lambda_{\bar{M}S}$ using Igi–Ono (type I and II) potential (in MeV).

		$\Lambda_{\bar{M}S}$				
States	Refs. 6, 24	100	200	300	400	500
$b = 20$[a]	$\alpha_s =$	0.1985	0.217	0.238	0.250	0.262
$1S$	6327	6329	6318	6310	6316	6327
$2S$	6906	6915	6904	6881	6880	6901
$3S$	7246	7264	7242	7244	7241	7252
$4S$		7508	7522	7545	7542	7552
$1P$	6754	6755	6744	6733	6732	6742
$2P$	7154	7144	7131	7125	7122	7134
$1D$	7028	7029	7017	7004	7000	7010
$2D$	7367	7334	7327	7327	7323	7333
$b = 5$[b]	$\alpha_s =$	0.1985	0.227	0.230	0.2405	
$1S$	6327	6331	6324	6316	6307	
$2S$	6906	6914	6898	6910[c]	6918	
$3S$	7246	7258	7277	7236	7201[c]	
$4S$		7521	7517	7478	7500	
$1P$	6754	6756	6743	6737	6730	
$2P$	7154	7142	7138	7134	7120	
$1D$	7028	7029	7015	7012[c]	7007	
$2D$	7367	7335	7323[c]	7314	7316	

[a] $c_0 = -0.022$ to -0.031 MeV.
[b] $c_0 = -0.019$ to -0.026 MeV.
[c] Carried out to the second correction order.

method with b = 19 and $_{\bar{M}S}$ = 300 MeV and also 390 MeV, and then b = 16.3 and $_{\bar{M}S}$ = 300 MeV which seems to be more convenient than $_{\bar{M}S}$ = 500 MeV used by other authors.[13] Results of this study are also presented in Table 6. It is clear that the overall study seems likely to be good and the reproduced masses of states are also reasonable. We see that the quark masses m_c and m_b are sensitive to the variation of $_{\bar{M}S}$. Therefore, as $_{\bar{M}S}$ increases the contribution of the potential (cf. e.g., Eqs. (40) and (41)) and consequently the binding energy $E_{n,l}$ term decreases which leads to an increase in the constituent quark masses of the convenient meson, cf. Eq. (18).

It is also found that the Qq potentials can reproduce the experimental masses of the bc states for various values of $_{\bar{M}S}$. Using this model, we see that the experimental bc splittings can be reproduced for $_{\bar{M}S} \sim$ 300 MeV in type I, $_{\bar{M}S} \sim$ 400 MeV in type II (cf. Table 5) and $_{\bar{M}S} \sim$ 300 MeV in type III (cf. Table 6). We also predicted the splittings in exact agreement with several MeV with the other formalisms (cf. Table 1 of Ref. 6).

In Table 6, we nd that m_c and m_b are insensitive to the variation of $_{\bar{M}S}$ for this Chen{Kuang (CK) potential. This is consistent with the conventional idea that, for

Table 5. The $\bar{b}c$ mass spectrum, splittings and leptonic constant predicted for various $\Lambda_{\bar{M}S}$ using Igi–Ono (type I and II) potential (in MeV).

	$\Lambda_{\bar{M}S}$				
States	100	200	300	400	500
Type I					
1^3S_1	6343	6334	6327	6334	6344
1^1S_0	6287	6272	6259	6263	6274
Δ_{1S}	56.3	62.0	68.3	71.1	69.8
$\|R_{1S}(0)\|^2$	0.826	1.005	1.156	1.19	1.114
$f_{B_c}^{\mathrm{NR}}$	354.1	391.1	420.0	426.0	411.7
$f_{B_c}^{(1\text{-loop})}$	328.1	356.5	376.8	379.2	364.4
$f_{B_c}^{(2\text{-loop})}$	290.0	306.1	311.7	306.5	287.1
$f_{B_c^*}^{\mathrm{NR}}$	352.6	389.2	417.7	423.6	409.4
$f_{B_c^*}^{(1\text{-loop})}$	326.7	354.8	374.7	377.1	362.4
$f_{B_c^*}^{(2\text{-loop})}$	288.7	304.6	310.0	304.7	285.6
Type II					
1^3S_1	6345	6340	6331	6323	
1^1S_0	6288	6279	6269	6259	
Δ_{1S}	56.7	60.6	61.8	64.4	
$\|R_{1S}(0)\|^2$	0.819	0.891	1.03	1.204	
$f_{B_c}^{\mathrm{NR}}$	352.7	368.2	396.0	428.6	
$f_{B_c}^{(1\text{-loop})}$	327.1	334.9	357.4	382.1	
$f_{B_c}^{(2\text{-loop})}$	289.1	283.0	300.1	314.4	
$f_{B_c^*}^{\mathrm{NR}}$	351.2	366.4	394.1	426.4	
$f_{B_c^*}^{(1\text{-loop})}$	325.6	333.3	355.7	380.1	
$f_{B_c^*}^{(2\text{-loop})}$	287.8	281.7	298.6	312.8	

heavy quarks, the constituent quark mass is close to the current quark mass which is $_{\bar{M}S}$ independent. Numerical calculations show that this potential is insensitive to $_{\bar{M}S}$ in the range from 100 to 300 MeV, and as $_{\bar{M}S}$ increases, the potential becomes more sensetive for the 1S-state only. The obtained n$^1S_0$ and n$^3S_1$ hyper ne splittings for the B_c meson in the Chen{Kuang potential are also listed in Table 6. They are considerably smaller than the corresponding values $_{1S}(bc)$ = 76 MeV, and $_{2S}(bc)$ = 42 MeV predicted by the quadratic formalism of Ref. 6. Moreover, Chen{Kuang[27] predicted $_{1S}(bc)$ = 49.9 MeV, and $_{2S}(bc)$ = 29.4 MeV for their potentials with $_{\bar{M}S}$ = 200 MeV in which the last splitting is almost constant as $_{\bar{M}S}$ increases. Our predictions for $_{1S}(bc)$ = 68 MeV, and $_{2S}(bc)$ = 35 MeV for the Chen{Kuang potential with $_{\bar{M}S}$ running from 100 into 375 MeV. We also nd $_{1S}(bc)$ = 67 MeV, and $_{2S}(bc)$ = 33 MeV for the Igi{Ono potential with

Table 6. The $\bar{b}c$ mass spectrum, splittings and leptonic constant predicted for various $\Lambda_{\bar{MS}}$ using Igi–Ono (type III) and Chen–Kuang potentials (in MeV).

State	IO (III)			CK		
$b =$	16.3	19	19	5.1	5.1	5.1
$\Lambda_{\bar{MS}} =$	300	300	390	100–300	350	375
$\alpha_s =$	0.250	0.2505	0.2205	0.270	0.270	0.270
$1S$	6309	6309	6297	6324	6372	6354
$2S$	6880	6870	6877	6880	6880	6880
$3S$	7247	7236	7254	7258	6258	6258
$4S$	7553	7541	7563	7570	7570	7570
$1P$	6725	6721	6737	6723	6723	6723
$2P$	7124	7114	7135	7127	7127	7127
$3P$	7441	7429	7452	7452	7452	7452
$1D$	6997	6990	7013	6993	6993	6993
$2D$	7328	7317	7341	7332	7332	7332
$3D$	7613	7599	7624	7625	7625	
1^3S_1	6326	6327	6315	6341	6389	6371
1^1S_0	6259	6258	6243	6273	6321	6304
Δ_{1S}	67.3	68.6	72.6	67.8	67.8	67.7
$\lvert R_{1S}(0)\rvert^2$	1.115	1.119	1.339	1.017	1.017	1.017
$f_{B_c}^{\mathrm{NR}}$	412. loop) $_{B_c}$	367. loop) $_{B_c}$	296. -26028.9123 0 Td (294)Tj /R35 7.97011 Tf 12.5768 0 Td (:)			

Table 7. The nS-levels leptonic constant of the $\bar{b}c$ 4fsystem, calculated in different static potential models (the accuracy is 3–7%), in MeV, using the SR.

Quantity	Cornell	Song–Lin	Turin	Martin	Logarithmic
f_{1S}	449.6	450.4	448.0	448.8	420.9
f_{2S}	305.8	305.0	303.3	303.5	284.7
f_{3S}	243.0	243.2	241.3	241.8	227.2
f_{4S}	206.0	207.1	204.9	205.9	193.8

The scaling relation (SR) for the S-wave heavy quarkonia has the form[25]

$$\frac{f_n^2}{M_n(bc)}\left(\frac{M_n(bc)}{M_1(bc)}\right)^2\left(\frac{m_c+m_b}{4\mu}\right)=\frac{d}{n}\,, \tag{69}$$

where m_c and m_b are the masses of heavy quarks composing of the B_c-meson, μ is the reduced mass of quarks, and d is a constant independent of both the quark avors and the level number n. The value of d is determined by the splitting between the 2S and 1S levels or the average kinetic energy of heavy quarks, which is independent of the quark avors and n with the accuracy accepted. The accuracy depends on the heavy quark masses and it is discussed in detail.[25] The parameter value in Eq. (69), $d \simeq 55$ MeV, can be extracted from the experimentally known leptonic constants ψ and . So, from Table 1, the SR gives the 1S-level

$$f_{B_c}^{(\mathrm{SR})} \simeq 444^{+6}_{-23}\ \mathrm{MeV} \tag{70}$$

for all static potentials used. Furthermore, Kiselev[25,32] estimated $f_{B_c} = 400 \pm 45$ MeV and $f_{B_c}^{(\mathrm{SR})} = 385 \pm 25$ MeV, Narison[42] found $f_{B_c}^{(\mathrm{SR})} = 400 \pm 25$ MeV, and also the optimal result of Chabab[43] was $f_{B_c} = 300 \pm 65$ MeV obtained by using two versions of QCD sum rules which took into account the uncertainties due to the variations of the continuum threshold within the stability regions.

On the other hand, we present the leptonic constants for the excited nS-levels of the bc in Table 7. We see that our prediction $f_{B_c(2S)}^{(\mathrm{SR})} = 300 \pm 15$ MeV is in good agreement with the ones predicted by Kiselev *et al.*,[18] $f_{B_c(2S)}^{(\mathrm{SR})} = 280 \pm 50$ MeV for the 2S-level in the bc system. This also agrees with the scaling relation.[25]

We conclude that the approximated values of the excited nS-states agree well with the simple scaling relation (SR) derived from QCD sum rules for the state density. It is clear that the estimates obtained from the potential model and SR are in good agreement with several MeV. However, the di erence between the leptonic constants for the pseudoscalar and vector 1S-states is caused by the spin-dependent corrections, which are small. Numerically, we get $|f_{B_c^*} - f_{B_c}|/f_{B_c^*} < 1\%$. For the heavy quarkonia, the QCD sum rule approximation provides that the f_P and f_V values for the pseudoscalar and vector states. Leptonic constant is practically independent of the total spin of quarks, so that

$$f_{V,n} \simeq f_{P,n} = f_n\,. \tag{71}$$

Our numerical approximation for the decay constants of the pseudoscalar and vector states in Tables 3, 5, and 6 is a con rmation to the last formula Eq. (71).

In this paper, we have developed the SLET in the treatment of the bc system using group of static and QCD-motivated potentials. For such potentials the method looks quite attractive as it yields highly accurate results. The convergence of this method seems to be very fast as the higher corrections to energy have lower contribution. In this context, in reproducing the SAD, we used the same tted parameters of the other authors, cf. Ref. 11 and the references therein, for the sake of comparison and testing the accuracy of our approach. Once the experimental leptonic constant of the B_c-meson becomes clear, one can sharpen the analysis.

Here, we would like to make the following general remark regarding the SLET: It is worthwhile to notice that the objectives of using the same wide class of quarkonium potentials with the same tting parameters in our previous work is to demonstrate to the readers that the SLET method generates exactly the same results as in the SLNET. It also refutes the claims of the authors in Ref. 15 that this method is a reformation of SLNET and has a wider domain of applicability. Therefore, it is just a simpler alternative parallel mathematical pseudoperturbative expansion technique.

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Appendix A. SLET Parameters for the Schrödinger Equation

Here, we list the analytic expressions of $\gamma^{(1)}$, $\gamma^{(2)}$, ε_i and δ_j for the Schrödinger equation:

$$\gamma^{(1)} = [(1 + 2n_r)\varepsilon_2 + 3(1 + 2n_r + 2n_r^2)\varepsilon_4] - \omega^{-1}[\varepsilon_1^2 + 6(1 + 2n_r)\varepsilon_1\varepsilon_3 + (11 + 30n_r + 30n_r^2)\varepsilon_3^2], \tag{A.1}$$

$$\begin{aligned}\gamma^{(2)} = {} & [(1 + 2n_r)\delta_2 + 3(1 + 2n_r + 2n_r^2)\delta_4 + 5(3 + 8n_r + 6n_r^2 + 4n_r^3)\delta_6 \\ & - \omega^{-1}(1 + 2n_r)\varepsilon_2^2 + 12(1 + 2n_r + 2n_r^2)\varepsilon_2\varepsilon_4 + 2\varepsilon_1\delta_1 \\ & + 2(21 + 59n_r + 51n_r^2 + 34n_r^3)\varepsilon_4^2 + 6(1 + 2n_r)\varepsilon_1\delta_3 \\ & + 30(1 + 2n_r + 2n_r^2)\varepsilon_1\delta_5 + 2(11 + 30n_r + 30n_r^2)\varepsilon_3\delta_3 \\ & + 10(13 + 40n_r + 42n_r^2 + 28n_r^3)\varepsilon_3\delta_5 + 6(1 + 2n_r)\varepsilon_3\delta_1] \\ & + \omega^{-2}[4\varepsilon_1^2\varepsilon_2 + 36(1 + 2n_r)\varepsilon_1\varepsilon_2\varepsilon_3 + 8(11 + 30n_r + 30n_r^2)\varepsilon_2\varepsilon_3^2 \\ & + 24(1 + 2n_r)\varepsilon_1^2\varepsilon_4 + 8(31 + 78n_r + 78n_r^2)\varepsilon_1\varepsilon_3\varepsilon_4 \\ & + 12(57 + 189n_r + 225n_r^2 + 150n_r^3)\varepsilon_3^2\varepsilon_4]\end{aligned}$$

$$-\omega^{-3}[8\varepsilon_1^3\varepsilon_3 + 108(1+2n_r)\varepsilon_1^2\varepsilon_3^2 + 48(11+30n_r+30n_r^2)\varepsilon_1\varepsilon_3^3 + 30(31+109n_r+141n_r^2+94n_r^3)\varepsilon_3^4], \quad (A.2)$$

where

$$\varepsilon_i = \frac{\varepsilon_i}{(4\mu\omega)^{i/2}}, \qquad i = 1, 2, 3, 4, \quad (A.3)$$

and

$$\delta_j = \frac{\delta_j}{(4\mu\omega)^{j/2}}, \qquad j = 1, 2, 3, 4, 5, 6, \quad (A.4)$$

$$\varepsilon_1 = \frac{-(2a+1)}{2\mu}, \qquad \varepsilon_2 = \frac{3(2a+1)}{4\mu}, \quad (A.5)$$

$$\varepsilon_3 = -\frac{1}{\mu} + \frac{r_0^5 V'''(r_0)}{6Q}, \qquad \varepsilon_4 = \frac{5}{4\mu} + \frac{r_0^6 V''''(r_0)}{24Q}, \quad (A.6)$$

$$\delta_1 = -\frac{a(a+1)}{2\mu}, \qquad \delta_2 = \frac{3a(a+1)}{4\mu}, \quad (A.7)$$

$$\delta_3 = -\frac{(2a+1)}{\mu}, \qquad \delta_4 = \frac{5(2a+1)}{4\mu}, \quad (A.8)$$

$$\delta_5 = -\frac{3}{2\mu} + \frac{r_0^7 V'''''(r_0)}{120Q}, \qquad \delta_6 = \frac{7}{4\mu} + \frac{r_0^8 V''''''(r_0)}{720Q}. \quad (A.9)$$

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