Decay of the Charged Harmonic Oscillator

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ABSTRACT

Abstract: We apply the van Kampen-Steinwedel principal axis transformation to the problem of finding time-dependent prob ability distributions for decaying states of the harmonic oscillator.
The result lends itself easily to computer simulation. Results for the decay of levels 1,2, and 15 are given.

(انخماد المذبذب التوافقي المشحون)

ملخص

نطبق تحويلات Van Kampen-Steinwedel للمحاور الرئيسية على مشكلة ايجاد التوزيع الاحتمالي الزمني للحالات المتخامدة للمذبذب . النتائج تصنع نفسها بسهولة بواسطة جهاز الحاسوب . نتائج لانخماد المستويات ١ ، ٢ و ١٥ معطاه.

I. Introduction

It is well known that the problem of the interaction of an elastically bound electron with quantized radiation field can be solved exactly within the dipole approximation. The problem was discussed long ago by Van Kampen[1), and by steinwedel[2). The problem reduces to one of finding normal modes of an oscillating system. Van Kampen derived a system of normal coordinates that permit one to find time dependent probability distributions for the electron. These distributions collapse in time to the ground state distribution (altered by the electromagnetic field's vacuum fluctuations).

These solutions for the electron's probability distribution were studied over thirty years ago by one of us(W.C.H)[3). At that time , computers were not readily available. It is therefore worthwhile to reexamine the problem and present new results.

In the van Kampen-Steinwedel model, the electron is elastically bound to the origin of the coordinate system by a spring of constant MK^2 , and the radiation is confined to a sphere of radius L. The boundary conditions are $\vec{A}(L) = 0$; $\vec{E}(L) = 0$. We set $\hbar = c = 1$ in this work.

In general, it is necessary to formulate the problem in terms of the normal coordinates. The canonical transformation has no corresponding unitary transformation in quantum theory. Any attempt to formulate intial conditions in the quantum problem in terms of the original particle and field coordanates leads to divergences. Representations of communation relations in the new and old coordinate systems are not unitarily equivalent. Inequivalent representations were first noticed by van Hove[4) and by Friedricks[5). A comprensive discussion of this general topic has been given by Haag[6).

The transformed Hamiltonian has the simple form

$$H = \frac{1}{2}(\vec{P}^2 + K^2\vec{R}^2) + \frac{1}{2}\sum(\vec{P}_n^2 + k_n^2\vec{a}_n^2) \qquad (1)$$

The old variables are then recoverd from the new variables by means of the transformation

$$\vec{a} = T\sum_n \vec{p}_n u_n + T\sqrt{\frac{3}{L_k}}\frac{\cos Kr}{Kr}\vec{P}'$$

$$\vec{p} = -\frac{1}{4\pi} T\sum_n \vec{a}_n k_n^2 u_n - \frac{1}{4}\pi T\sqrt{\frac{3}{L_k}}\frac{\cos Kr}{r} K\vec{R}'$$

$$\vec{R} = \sqrt{\frac{3}{L_k}}\frac{1}{eK}\vec{R}' - \sum_n \sqrt{\frac{3}{L}}\frac{\sin \eta_n}{ek_n}\vec{a}_n$$

$$\vec{P} = \sqrt{\frac{3}{L_k}}\frac{M}{eK}\vec{P}' - \sum_n \sqrt{\frac{3}{L}}\frac{MK^2}{e}\frac{\sin \eta_n}{k_n^3}\vec{p}_n$$

(2)

In the above a is the vector potential in the Coulomb gauge and $\vec{P} = \frac{1}{4\pi}\frac{\partial \vec{a}}{\partial t} = -\frac{1}{4\pi}\vec{E}$, where $\vec{E}$ is the transverse electric field. $\vec{R}$ is the electron displacement and $\vec{P}$ is the electron's canonical momentum. The functions $u_n(r)$ are given by

$$u_n(r) = \sqrt{\frac{3}{L}}\frac{\sin(k_n r - \eta_n)}{k_n r}$$

(3)

and L_k is given by

$$L_k = L + \frac{3M}{e^2 K^2}$$

(4)

In the first two equations of (2), the symbol T means that the transverse part of the vector field is to be taken. The boundary conditions at r = L imply that

$$k_n L - \eta_n = n\pi\ ,\quad n = \text{ integer}$$

(5)

and the phase shifts are given by

$$\tan k_n L = \tan \eta_n = \frac{k_n}{\kappa}\frac{k_n^2}{k_n^2 - k^2}$$

(6)

with $\kappa = \frac{3M}{2e^2}$. For further details , the reader is referred to refs. 2 and 3.

II. Review of previous results

For completeness, we first collect relevant results of reference 3.

The width of the ground state wave packet was shown in ref. 3 to be

$$< Z^2 >_{Gs} = \frac{1}{2MK}[1+\frac{2K}{\pi\kappa}(\ln\frac{\kappa}{K}-\frac{1}{2})] \tag{7}$$

Eq. (7) shows the spreading of the packet due to the vacuum fluctuations of the electromagnetic field. In the following, we consider excitations of the oscillator in the Z direction (one dimensional excitation of a three dimensional oscillator).

We first consider the decay of the first excited state. The initial condition is formulated by demanding that the field energy be a minimum, that <Z> vanish, and that the state function be a superposition of single quantum states

$$|\phi>= \sum_{n=0}^{\infty} c_n |n> \tag{8}$$

The state $|n>$ is one in which the n^{th} oscillator is excited once in the Z direction. The conditions above yield the result

$$c_n = \frac{2}{3}\sqrt{\frac{3}{L}} e \cos\eta_n \sqrt{k_n} \frac{1}{\sqrt{KM}} \frac{K^2}{K^2-k_n^2}$$

$$c_0 = \frac{\sqrt{3M}}{L_k} \frac{1}{eK} \tag{9}$$

In eq. (9), the coefficient c_o is the one that goes with the primed oscillator of eq. (1).

Defining $\Delta <Z^2> = <Z^2>-<Z^2>_{G.s.}$, one finds

$$\Delta < Z^2 > (t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{3}{L} \frac{\sin \eta_m \sin \eta_n}{e^2 k_m k_n} \frac{c_m c_n}{\sqrt{k_m k_n}} \cos(k_m - k_n)t \tag{10}$$

In the limit $L \to \infty$, eq. (10) becomes

$$\Delta < Z^2 > (t) = \frac{1}{MK} e^{-\frac{K^2}{\kappa} t} \tag{11}$$

The decay of the N^{th} excited state may be treated in a similar way. Let $\alpha^+_{n_k}$ and α_{n_k} be creation and destruction operators corresponding to the n_kth oscillator . An-N-quantum eigenstate of the Hamiltonian of eq. (1) is then $|N, n_1, \ldots, n_N >$

Then , as usual , we have

$$\alpha^+_{n_k} |N, n_1 \ldots , n_N > = \sqrt{N+1} |N+1, n_k, n_1, \ldots n_N >$$

$$\alpha_{n_k} |N, n_1, \ldots , n_N > = \frac{1}{\sqrt{N}} \sum_i \delta_{n_k n_i} |N-1, n-1, \ldots n_{i-1}, n_{i+1}, \ldots n_N > \tag{12}$$

The superposition

$$|\phi > = \sum_{n_1} \sum_{n_N} c_{n_1 \ldots n_N} |N, n_{1, \ldots} n_N >, \tag{13}$$

with

$$c_{n_1 \ldots n_N} = \prod_{n_k = n_1}^{n_N} c_{n_k} \tag{14}$$

where the c_{nk} are given by eq (9), then yields

$$\Delta < Z^2 > (t) = \frac{N e^{-K^2/kt}}{MK} \tag{15}$$

This result illustrates that the oscillator is strictly linear, with the lifetime independent of the level of excitation.

The result given thus far those of ref. 3. The point of this paper is that , since the intial state of the decaying system is specified, one should be able to construct the probability distribution for the electron at arbitrary times , and in this way create computer simulations of decaying states of the oscillator.

Because of eq. 7, we shall henceforth define the coordinate Z in units of $\sqrt{1+\frac{2K}{\pi\kappa}(\ln\frac{n}{K}-\frac{1}{2})}\,/\,\sqrt{MK}$. The so-defined coordinate is just the usual dimensionless coordinate for one dimensional oscillator augmented by a term due to vacuum fluctuations. Thus we define

$$\xi=\frac{\sqrt{MK}}{\sqrt{1+\frac{2K}{\pi\kappa}(\ln\frac{\kappa}{K}-\frac{1}{2})}}Z \tag{16}$$

III. Decay of the N^{th} excited state of the oscillator

Our task consists of establishing a correspondence between the modes of the distribution implied by equations 9,13 and 14. The fact that the coefficients are expressed as a product (eq. 14), indicates that we can form modes of the distributions of n quantum states for which the field energy is minimum. The modes of these distributions may be written

$$<n|\xi^m|n>=B(m,n)<0|\xi^2|0>=\left(\frac{1}{2}\right)^{m/2}B(m,n) \tag{17}$$

where B(m,n) is to be specified.

Eq. 7 for example , gives $<0|\xi^2|0>=\frac{1}{2}$. Eq. 11 yields (at τ =0), $<1|\xi^2|1>=\frac{3}{2}$

Because of the product structure of the coefficients in the superposition, we compute modes of the ditribution by simply counting graphs which are constructed as follows : From a horizontal base line show n incoming lines (quanta of intial state) and n outgoing lines (quanta of final state), as in fig. 1 . In order to form $<n|\ \xi^m|n>$, one needs graphs having m vertices, with the number of free lines (necessarily even!) between 0 and 2n> The graph has m vertices, so that for q lines $\frac{m-q}{2}$, is the number of bubbles. The bubbles represent emission and subsequent absorption of agiven quantum. Eq. 7 is the result of graph consisiting of a single bubble. Fig. 1 illustrates a typical graph, for which m=8 and q=2, which has three bubbles.

Let us count graphs with m vertices and $\frac{m-q}{2}$ bubbles. Then $\frac{m!}{(\frac{m-q}{2})!2^{\frac{m-q}{2}}}$ is the number of ways of distributing the bubbles among the m vertics. The factor $2^{\frac{m-q}{2}}$ occurs because interchanging the ends of each bubble leads to the same diagram.

The diagram involved in $<n|\ \xi^m|n>$ have q free lines, where q(even)varies from 0 to 2n . These "scattering state photons" are associated with different frequencies. We have q/2 incoming lines (absorbed photons) and q/2 exit lines (emitted photons). The number of ways of picking q photons from the n original ones is

$$n(n-1)\ldots(n-\frac{q}{n}+1)=\frac{n!}{(n-q/2)!}$$

We must have the same incoming frequencies as out going ones. Morover, there are $\left(\frac{q}{2}\right)!$ permutation of the incoming photons and $\left(\frac{q}{2}\right)!$ permutations of the outgoing ones.

This yields

$$B(m,n)=\sum_{q=0}^{\min m,\,2n}\frac{m!\,n!}{\left(\frac{m-q}{2}\right)!\,2^{\frac{m-q}{2}}\left(n-\frac{q}{2}\right)!\left[\left(\frac{q}{2}\right)!\right]^2} \qquad (18)$$

Since q cannot exceed m, the upper limit on the sum must be the minimum of m and 2n. It must now be show that the modes of the distribution correspond (after taking of avery small broading due to vacuum fluctuations) to the modes of harmonic oscillator wave functions. We must evaluate

$$\int_{-\infty}^{\infty}\xi^{m}[H_m(\xi)]^2e^{-\xi^2}d\xi$$

We began with the identity[7].

$$\sum_m\sum_k\frac{S^m t^k}{mk}\int_{-\infty}^{\infty}H_m(\xi)H_k(\xi)e^{2\lambda\xi-\xi^2}d\xi=\sqrt{\pi}e^{\lambda^2+2(St+\lambda S+\lambda t)} \qquad (19)$$

We operate on both sides with $\left(\frac{\partial}{\partial\lambda}\right)^m$ and then put $\lambda = 0$, and obtain

$$\sum_m\sum_K \frac{S^m t^k 2^m}{n!k!}\int_{-\infty}^{\infty} \xi^m H_n(\xi)H_K(\xi)e^{-\xi^2}d\xi$$

$$= \lim_{\lambda\to 0} = \frac{\partial^m}{\partial\lambda^m}\{\sqrt{\pi}e^{2st}e^{\lambda^2+(2s+2t)\lambda}\}$$

$$= \lim_{\lambda\to 0}\sqrt{\pi}e^{2st}\frac{\partial^m}{\partial\lambda^m}\{e^{(\lambda+s+t)^2}e^{-(s+t)^2}\} \qquad (20)$$

$$= \sqrt{\pi}\lim_{\lambda\to 0}e^{-(s^2+t^2)}\frac{\partial^m}{\partial\lambda^m}e^{(\lambda+s+t)^2} = \sqrt{\pi}e^{-(s^2+t^2)}\frac{\partial^m}{\partial s^m}e^{(s+t)^2}$$

$$= \sqrt{\pi}e^{2st}e^{-(s+t)^2}\frac{\partial^m}{\partial s^m}e^{(s+t)^2} = \sqrt{\pi}e^{2st}i^m e^{\xi^2}\frac{d^m}{d\xi^m}e^{-\xi^2}$$

with $\xi = i(s+t)$.

The last term of eq. (20) is $\sqrt{\pi}e^{2st}(-i)^m H_m(\xi)$. We must have

$$\sum_n\sum_k \frac{S^m t^k}{n!k!}2^m\int_{-\infty}^{\infty}\xi^m H_n(\xi)H_k(\xi)e^{-\xi^2}d\xi = \sqrt{\pi}e^{2st}(-i)^m H_m[i(s+t)] \qquad (21)$$

We are interested in the case n=k and m even. We note that $(-i)^m H_m(\xi)$ for ξ pure imaginary gives just $H_m(\mathrm{Im}\xi)$ with the modification that all terms have a positive sign . Therefore, $(-i)^m$ $H_m(ix)$ is a positive definite polynomial of degree m for m even and x real. This polynomial is[8] $\sum_{q=0}^{m} a_q(m)x^q$, with

$$a_q(m) = \frac{m!2^q}{q!(\frac{m-q}{2})!} . \qquad (22)$$

Taking the n= k terms of eq. (21) for m even , we have

$$\sum_n \frac{(st)^n}{(n!)^2} 2^m \int_{-\infty}^{\infty} \xi^m [H_m(\xi)]^2 e^{-\xi^2} d\xi = \sqrt{\pi} e^{2st} \sum_{q=0}^{m} \frac{a_q(m) q!}{[(q/2)!]^2} \sqrt{st^q}$$

$$= \sqrt{\pi} \left(\sum_{i=0}^{\infty} \frac{(2st)^i}{i!} \right) \sum_{q=0}^{m} \frac{q!}{[(\frac{q}{2})!]} (st)^{q/2} a_q(m)$$

(23)

Since $q! = 2^{q/2} (\frac{q}{2})!(q-1)!!$, we have $\frac{q!}{(q/2)!} = 2^{q/2}(q-1)!!$.

Equating powers of st in eq. (23) yield $i + \frac{q}{2} = n$, so that $i = n - \frac{q}{2}$, with the restriction that i is non-negative . We now have

$$\frac{2^m}{(n!)^2} \int_{-\infty}^{\infty} \xi^m [H_n(\xi)]^2 d\xi = \sqrt{\pi} \sum_{\substack{q=0 \\ q\,\text{even}}}^{\text{Min}(m,2n)} \frac{2^{n-q/2} 2^{q/2}}{(n-q/2)!(q/2)!} (q-1)!! a_q(m)$$

(24)

We finally obtain

$$\int_{-\infty}^{\infty} \xi^m [H_n(\xi)]^2 e^{-\xi^2} d\xi = \sqrt{\pi} 2^{n-m} n! \sum_{q=0}^{\text{Min}(m,2n)} \binom{n}{\frac{q}{2}} (q-1)!! a_q(m)$$

$$= \sqrt{\pi} 2^{n-m} n! \sum_{\substack{q=0 \\ q\,\text{even}}}^{\text{Min}(m,2n)} \frac{n!(q-1)!}{(n-\frac{q}{2})! q! (\frac{m-q}{2})!} m!\, 2^q \qquad (25)$$

$$= \sqrt{\pi} 2^{n-m} (n!)^2 \sum_{\substack{q=0 \\ q\,\text{even}}}^{\text{Min}(m,2n)} \frac{m! 2^{q/2}}{(n-q/2)!(\frac{m-q}{2})![(q/2)!]^2}$$

Normalization of states yields

$$<n|n>=\frac{2^{-n}}{n!\sqrt{\pi}}\int_{-\infty}^{\infty}[H_N(\xi)]^2 e^{\xi^2}d\xi \ , \tag{26}$$

so that our final result is

$$<n|\xi^m|n>=2^m n!\sum_{\substack{q=0\\ q\,even}}^{Min(m,2n)}\frac{m!2^{q/2}}{(n-q/2)!(\frac{m-q}{2})![(q/2)!]^2} \tag{27}$$

The upper limit of the sum in eqs. 24-27 is the smaller of m and 2n, since the arguments of the denominator must be non-negative. Equation (27) is exactly the result of eqs. (17) and (18). This proves that, indeed, our initial conditions of eqns. (9), (13) and (14) yield probability distribution at t=0 that are harmonic oscillator state wave functions that have been modified to account for vacuum fluctuations.

We are now in a position to discuss the decay of the distribution. We see from the discussion in ref. 3 that every pair of incoming and outgoing photon lines gives rise to a factor $e^{-\gamma\tau}$ in the distribution, where $\gamma=\frac{K^2}{K}$

Consider an arbitrary mode for n=1:

$$<1|\xi^m|1>(m+1)!!<0|\xi^2|0>^{m/2} \quad \text{at } t=0$$

Then

$$<1|\xi^m|1>(t)=\{[B(m,1)-B(m,0)]e^{-\gamma t}+B(m,0)\}<0|\xi^2|0>^{m/2} \tag{28}$$

$$P(\xi,t) = \left[|\psi_1(\xi)|^2 - |\psi_0(\xi)|^2 e^{-\gamma t} + |\psi_0(\xi)|^2\right. \tag{29}$$

Here we use $|\psi_1(\xi)|^2$ to denote the probability distribution for the first excited state of the oscillator. For our system, the function $\psi(\xi, t)$ does not exist, but the notation is useful in describing the distribution. We note that $\int_{-\infty}^{\infty} P(\xi,t)d\xi = 1$ for all times, and that the distribution given in eq. 28 gives the proper values at $t = 0$ and $t = \infty$. We may now generalize to our ultimate result, which is that the nth excited state decays as

$$P_n(\xi,t) = [|\psi_n(\xi)|^2 - |\psi_{n-1}(\xi)|^2]e^{-n\gamma t} + [|\psi_{n-1}(\xi)|^2 - |\psi_{n-2}(\xi)|^2]e^{-(n-1)\gamma t} + \ldots\ldots + [|\psi_1(\xi)|^2 - |\psi_0(\xi)|^2]\, e^{-\gamma t} + |\psi_0(\xi)|^2. \tag{30}$$

IV. Results and discussion

Eq. (30) enables one to easily produce a computer simulation of the decay of oscillator states. Figures 2, 3 and 4 gives some typical results. Fig. 2 shows the first excited state of the oscillator at time γt=0,0.3,1,and 20. Fig. 3 discribes the decay of the second excited state at time γt= 0, 0.3, 0.6, 20. The value $\gamma t = 20$ was chosen to illustrate that, after times much longer than t = 1/2, the oscillator is in its ground state.

Fig. 4, which illustrates the decay of a more highly excited state, shows the decay of the n = 15 level at times γt = 0, 0.1, 0.5, 1.0. It is interesting to note that the initial ripples in the distribution have almost vanished after 0.1 lifetimes. By half the lifetime, the distribution already appears to be Gussian. At one lifetime, the Gussian has narrowed.

Finally, we note that one can obtain the result given here using the Heisenberg-Langevin approach[9)]. However, the present method must be regarded as more rigorous, since it does not make use of the Wigner-Weisskopf approximation.

The reader is cautioned that equation (30) applies only to the harmonic oscillator, which is a strictly linear system. Multiphoton transitions do not occur in the case of the oscillator. Eq. (30) shows that the decay of the higher state evolves in a series of steps in which the excitation level drops by one unit of $\hbar k$.

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Figure Captions

Fig. 1: Graph for m = 8, q = 2.

Fig. 2: Decay of 1st excited state; $\gamma t = 0, 0.3, 1, 20$.

Fig. 3: Decay of second excited state; $\gamma t = 0, 0.3, 0.6, 20$.

Fig. 4: Deacy of 15th excited state; $\gamma t = 0, 0.1, 0.5, 1.0$.

Fig. 1 Graph for $m = 8, q = 2$.

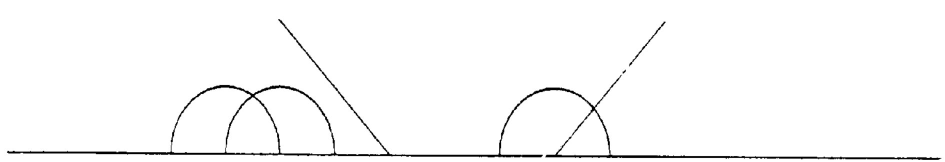
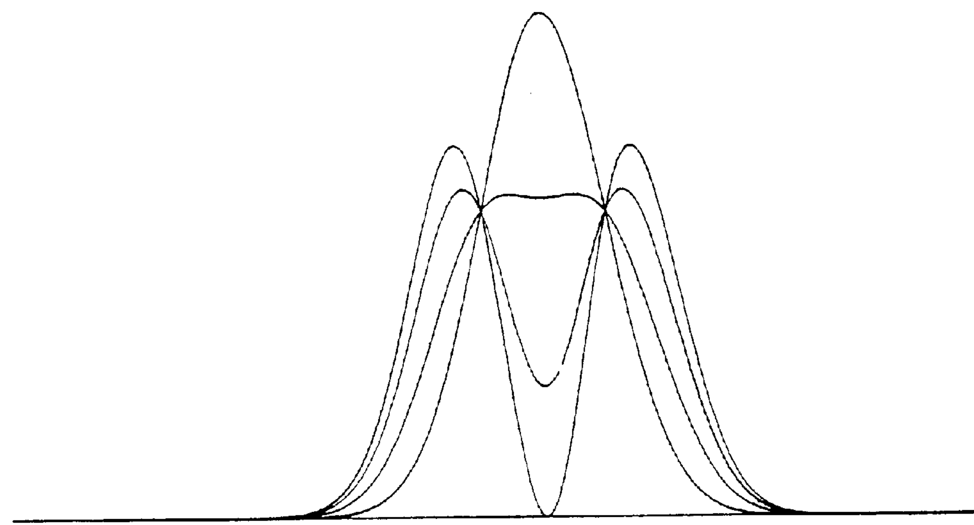


Fig. 2 Decay of 1st excited state; $\gamma t = 0, = .3, 1, 20.$



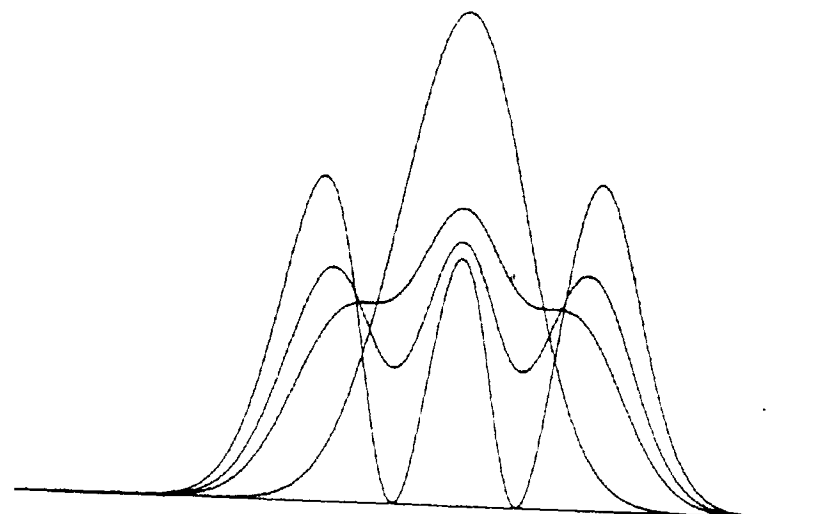


Fig. 3 Decay of second excited state;yt =0,0.3,0.6,20.

Fig. 4 Deacy of 15th excited state;yt = 0,0.1,0.5,1.0.