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Multidimensional electrostatic energy and classical renormalization

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ABSTRACT

Recent interest in problems in higher space dimensions is becoming increasingly important and attracted the attention of many investigators in variety of fields in physics. In this paper, the electrostatic energy of two geometries (a charged spherical shell and a non-conducting sphere) is calculated in higher space dimension, *N*. It is shown that as the space dimension increases, up to *N* = 9, the electrostatic energy of the two geometries decreases and beyond *N* = 9 it increases. Furthermore, we discuss a simple example which illustrates classical renormalization in electrostatics in higher dimensions.

Keywords: Electrostatic Energy; Higher Dimensions; Renormalization

1. INTRODUCTION

The space dimension N plays an important role in studying many physical problems. It has been used for the radial wave functions of the hydrogen like atoms in N dimensions [1,2]. Exactly solvable models have also been investigated [3,4]. In addition, a great deal of recent work in field theory [5], high energy physics [6], and in cosmology [7] has been conducted. Furthermore, problems of mathematical interest have been investigated in higher dimensions [8,9]. One of the fundamental quantities in physics is the electrostatic energy which is currently investigated by many workers in various areas [10-12]. Therefore, the present author is motivated to consider the effect of space dimension on the electrostatic energy of two simple, but illustrative, systems. A connected technique to electrostatic energy is the renormalization in classical field theory. Renormalization is needed to eliminate divergences which appear in the computation of Feyman graphs so that sensible physical results can be achieved [13-15]. Just recently, Corbò [16] considered renormalization technique in classical fields and Tort [17] discussed renormalization of electrostatic energy. So in the present paper, we will consider an example of classical renormalization of electrostatic energy in higher space dimensions. The organization of the present paper is as follows: In Section 2, we consider electrostatic energy in a hyper spherical shell. In Section 3, we calculate the electrostatic energy of a non conducting hyper sphere. In Section 4, we present an example of renormalization of electrostatic energy in higher space dimensions. Section 5 is devoted for conclusions.

2. ELECTROSTATIC ENERGY OF A HYPER SPHERICAL SHELL

We consider a charged hyper spherical shell of radius R and charge Q in N-dimensional space. Our purpose is to calculate the electrostatic energy of the shell by two methods. In the first method, we calculate the work done to bring the charge Q infinitesimally from infinity to the surface of the shell, while in the second method, we evaluate the volume integral over the squared of the electric filed, $\vec{E}$. The two methods require the electric field and the electric potential in space. Gauss's Law in N-dimensions is

$$\oint \vec{E}\cdot d\vec{A} = \oint E\, r^{N-1} d\Omega = \frac{q_{enc}}{\varepsilon_0} \tag{1}$$

The angular surface integral gives [18],

$$\oint d\Omega = \frac{2\pi^{N/2}}{\Gamma(N/2)}, \tag{2}$$

where $\Gamma(x)$ is the Gamma function. Since the charge is distributed on the surface, the above two equations yield

$$E = \frac{Q\,\Gamma(N/2)}{2\varepsilon_0 \pi^{N/2} r^{N-1}}, \quad r > R \tag{3}$$

and $E = 0$ for $r < R$. The electric potential is given by

$$V = -\int_{\infty}^{r} \vec{E} \cdot d\vec{r} = \frac{Q\,\Gamma(N/2)}{2\varepsilon_0 (N-2)\pi^{N/2} r^{N-2}}\ , \quad r \ge R \tag{4}$$

The first method yields the electrostatic energy, W as

$$W = \frac{1}{2}\int \sigma\ V\ dA = \frac{Q^2 \Gamma(N/2)}{4\varepsilon_0 \pi^{N/2}(N-2)} \frac{1}{R^{N-2}}\,, \tag{5}$$

Which can be written as

$$W = \frac{Q^2}{2\varepsilon_0 S (N-2) R^{N-2}}\,, \tag{6}$$

where σ is the surface charge density and S is the surface area of a unit shell as given in **Eq.2**. The second method enables us to write

$$W = \frac{1}{2}\varepsilon_0 \int E^2 d\tau = \frac{1}{2\varepsilon_0}\left(\frac{\Gamma(N/2)}{2\pi^{N/2}}\right)^2 \int_R^{\infty} \frac{Q^2}{r^{2N-2}} r^{N-1} dr d\Omega$$

$$= \frac{Q^2 \Gamma(N/2)}{4\varepsilon_0 \pi^{N/2}(N-2)} \frac{1}{R^{N-2}}\,, \tag{7}$$

Which is the same result given in **Eq.5**. It is interesting to note that our result yields the well-known result [19] for the three-dimensional case (N = 3), namely $W = \frac{Q^2}{8\pi\varepsilon_0 R}$. It is noticed that the electrostatic energy of the hyper shell depends on the space dimension N. It is illustrative to calculate the electrostatic energy (W_{shell}), with R = 1, for different values of N. This is calculated in units of $Q^2/8\pi\varepsilon_0$ and is shown in the second column of **Table 1**. Our calculated results clearly show that the electrostatic energy has a minimum at the space dimension N = 9. This can be explained as follows: In higher space dimensions, there are more orientations in space and thus more angles (N-1). This implies that it is relatively easy to assemble electric charges on the hyper surface of the shell which explains the decrease in the electrostatic energy as the space dimension increases up to N = 9. However, beyond this value of N, the surface area of the shell becomes smaller and smaller so that the decrease in the surface area, as N increases, dominates over the increase in the angular orientation. In mathematical terms, the surface area times (N-2) has a maximum at N = 9 and thus the electrostatic energy has a minimum at that value of N. It is tempting to investigate the behavior of the electrostatic energy for very large N. This can be checked by using Stirling's formula [20]

$$\Gamma(1+n) = n! \approx n^n \sqrt{2\pi n}\, e^{-n}\,, \tag{8}$$

and letting $n \to (N-2)/2$, one finds for very large N

$$W_{shell} \approx \frac{Q^2}{4\varepsilon_0}\left(\frac{1}{2\pi e}\right)^{(N-2)/2} (N-2)^{(N-3)/2}\,. \tag{9}$$

In the infinite dimensional space, the above equation gives an infinite electrostatic energy in the limit as $N \to \infty$. In this limiting case the surface area of the shell vanishes as can be seen from **Eq.2** and the use of Stirling's formula. Therefore, the shell behaves like a point charge in the infinite dimensional space and thus one expects the divergence of the electrostatic energy as an infinite self energy of a point particle.

3. ELECTROSTATIC ENERGY OF A CHARGED NON-CONDUCTING HPER SPHERE

Our main purpose here is to calculate the electrostatic energy of a uniformly charged non-conduction sphere in N-dimensional space. Following the second method of Section 2, we calculate the electric field inside and outside the sphere. The application of Gauss's Law given in **Eq.1** gives

$$\vec{E} = \frac{Q\Gamma(N/2)}{2\varepsilon_0 \pi^{N/2}} \begin{cases} \frac{1}{r^{N-1}}\hat{r} & r \ge R \\ \frac{r}{R^N}\hat{r} & r \le R\,, \end{cases} \tag{10}$$

where Q is the charge in the sphere. The electrostatic energy of the hyper sphere is thus

$$W_{sphere} = \frac{1}{2}\varepsilon_0 \int E^2 d\tau$$

$$= \frac{1}{2}\varepsilon_0 \left(\frac{Q\Gamma(N/2)}{2\varepsilon_0 \pi^{N/2}}\right)^2 \left[\int_R^{\infty} \frac{r^{N-1}}{r^{2N-2}} dr + \int_0^R \frac{r^{N+1}}{R^{2N}} dr\right] \int d\Omega\,.$$

The integrals in the curly bracket yield $\frac{2N}{(N^2-4)} \frac{1}{R^{N-2}}$ and the integral over Ω is given by **Eq.2**, Therefore, the electrostatic energy is simplified to

$$W_{sphere} = \frac{N\, Q^2\, \Gamma(N/2)}{2\varepsilon_0 \pi^{N/2} R^{N-2} (N^2-4)}\,, \tag{11}$$

which can be written as

$$W_{sphere} = \frac{Q^2}{\varepsilon_0 V (N^2-4) R^{N-2}}\,, \tag{12}$$

where $V(= 2\pi^{N/2}/N\Gamma(N/2))$ is the volume of the unit sphere in the N-dimensional space [18]. Clearly, the above electrostatic energy depends on the space dimension N, and it yields the well-known result [19] for N = 3, namely

$$W_{sphere}(N=3) = \frac{1}{4\pi\varepsilon_0} \frac{3Q^2}{5R}.$$

It is again constructive to calculate the electrostatic energy, in units of $Q^2/4\pi\varepsilon_0$, with R = 1 for different values of N. This is shown in the last column of **Table 1**. As before, the electrostatic energy has a minimum at the space dimension $N = 9$. But here, the volume of the hyper sphere time (N^2-4) has a maximum at $N = 9$ and hence the electrostatic energy has a minimum at that value. As it was checked in the previous section, the electrostatic energy becomes infinite in the infinite dimensional space ($N \to \infty$). In this limiting case the volume of the hyper sphere vanishes and thus the sphere behaves as a point charge with an infinite self energy.

4. RENORMALIZATION OF ELECTROSTATIC ENERGY

Renormalization, as is widely believed, is required in quantum field theory [21-23]. The main task of renormalization is to handle and eliminate the divergences so that one can obtain sensible physical results. Recently, it has been reported that renormalization can be applied to classical fields: For example, Corbò [16] gave two examples for renormalization of electrostatic potential and Tort [17] presented an example for renormalization of electrostatic energy. Our purpose here is to generalize Tort's example to higher space dimension *N*. Beside its mathematical interest, we will show that the divergence (or so-called singularity) of the electrostatic energy persists in the infinite dimensional space. Following Tort's model for the classical atom, we consider a point electric charge of magnitude *Ze*, where *Z* is the atomic number and *e* is the electron charge, surrounded by a concentric thin hyper-spherical shell of radius R and electric charge equal to $-Ze$. Ionization (partial or total) of this atom amounts to the removal of part of or the entire negative charge from the shell. This can be achieved by letting $-Ze \to -Ze(1-\lambda)$, where $\lambda \in [0,1]$. We will show below that the renormalization of the electrostatic energy (ΔU) in N dimensions is given by

$$\Delta U = U_{final} - U_{initial} = \frac{(\lambda Ze)^2 \Gamma(N/2)}{4\varepsilon_0 \pi^{N/2}(N-2)R^{N-2}} \tag{13}$$

The electric field inside the shell is only due to the point charge, since there is no contribution comes from the shell. Thus, the application of Gauss's Law, given in **Eq.1**, yields

$$\vec{E} = \frac{Ze\Gamma(N/2)}{2\varepsilon_0 \pi^{N/2} r^{N-1}}\hat{r}, \qquad 0 < r < R \tag{14}$$

and $\vec{E} = 0$ for $r > R$. The initial electrostatic energy before ionization can be calculated as

$$\begin{aligned} U_{initial} &= \frac{1}{2}\varepsilon_0 \int E^2 d^N r \\ &= \frac{1}{2\varepsilon_0}\left(\frac{Ze\Gamma(N/2)}{2\pi^{N/2}}\right)^2 \int_0^R \frac{1}{r^{2N-2}} r^{N-1} dr d\Omega \\ &= \frac{Z^2 e^2 \Gamma(N/2)}{4\varepsilon_0 \pi^{N/2}(N-2)}\left(-\frac{1}{r^{N-2}}\right)_0^R \end{aligned} \tag{15}$$

Obviously, the function $1/(r^{N-1})$ diverges at the origin and thus we have a singular point at $r = 0$. As Tort suggested, we can avoid this problem by introducing a finite non-null radius δ for the point charge and thus

$$U_{initial} = \frac{Z^2 e^2 \Gamma(N/2)}{4\varepsilon_0 \pi^{N/2}(N-2)}\left(\frac{1}{\delta^{N-2}} - \frac{1}{R^{N-2}}\right). \tag{16}$$

Now, when the atom is ionized part of the charge $(-\lambda Ze)$ of the shell will move to infinity and thus the enclosed charge within a hyper-spherical Gaussian surface of radius $r > R$. Will be $q = Ze - Ze(1-\lambda) = \lambda Ze.$ It is clear that the electric field, for $r > R$, remains the same as before ionization (see **Eq.14**) and for $r > R$ Gauss's Law immediately gives

$$\vec{E} = \frac{\lambda Ze\Gamma(N/2)}{2\varepsilon_0 \pi^{N/2} r^{N-1}}\hat{r}. \qquad r > R \tag{17}$$

Therefore the final electrostatic energy becomes

$$\begin{aligned} U_{final} &= \frac{Z^2 e^2 \Gamma(N/2)}{4\varepsilon_0 \pi^{N/2}(N-2)}\left(\frac{1}{\delta^{N-2}} - \frac{1}{R^{N-2}}\right) \\ &+ \frac{\varepsilon_0}{2}\int_R^\infty \left(\frac{\lambda Ze\Gamma(N/2)}{2\varepsilon_0 \pi^{N/2} r^{N-1}}\right)^2 r^{N-1} dr d\Omega \end{aligned} \tag{18}$$

The first term is just $U_{initial}$ and the integral in the second term has the same form as that of **Eq.7** and thus, one gets

$$U_{final} = U_{initial} + \frac{(\lambda Ze)^2 \Gamma(N/2)}{4\varepsilon_0 \pi^{N/2}(N-2)} \frac{1}{R^{(N-2)}} \tag{19}$$

Therefore, the change in the electrostatic energy is

$$\Delta U = U_{final} - U_{initial} = \frac{(\lambda Ze)^2 \Gamma(N/2)}{4_0 \pi^{N/2}(N-2)} \frac{1}{R^{N-2}}, \tag{20}$$

which is exactly the same as the electrostatic energy of a hyper-spherical shell that we found in Section 2. It is noticed that the variation of electrostatic energy is finite for all values of space dimension *N*, except for $N = \infty$ where ΔU becomes infinite. Therefore, the renormalization of the electrostatic energy works out for all space dimensions but failed in the infinite dimensional space. The persistent of the singularity in the infinite dimensional space is a result of the infinite electrostatic energy of the hyper-shell in that space, as we outlined in Section 2.

5. CONCLUSIONS

We have obtained the electrostatic energy of two systems (a charged spherical shell and a charged non-conducting sphere) in the *N*-dimensional space. Our calculated results show that the electrostatic energy decreases as the space dimension increases up to $N = 9$ and it increases without limit beyond that

Table 1. The electrostatic energy of the shell and the sphere as function of space dimension.

N	$W_{shell}(Q^2/8\pi\varepsilon_0)$	$W_{sphere}(Q^2/4\pi\varepsilon_0)$
3	1	0.6
4	0.318	0.2
5	0.159	0.11
6	0.101	0.07
7	0.076	0.06
8	0.065	0.051
9	0.060	0.049
10	0.062	0.051
11	0.067	0.057
12	0.078	0.067
13	0.097	0.08
14	0.125	0.11
15	0.169	0.15
16	0.238	0.21
17	0.349	0.31
18	0.531	0.47
19	0.835	0.75
20	1.353	1.23
30	6.8×10^3	640
40	2.3×10^6	2.18×10^6
50	3.0×10^{10}	2.9×10^{10}
100	5.4×10^{36}	5.3×10^{36}

value. This behavior is explained as follows: Each of the quantities $S(N-2)$ and $V(N^2-4)$ has a maximum at $N=9$ and thus the electrostatic energy of each system has a minimum at this value, as shown in **Eqs.6** and **12**. Our results also show that the electrostatic energy, for both systems, becomes infinite in the infinite dimensional space. Furthermore, we considered classical renormalization of electrostatic energy for a simplified model of a classical atom in higher space dimension. It was shown that the variation in electrostatic energy (the final minus the initial energy) is exactly the same as that of the hyper-shell, and thus the singularity persists in the infinite dimensional space.

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