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Some Aspects of an Infinite N-dimensional Spherical Potential Well

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Abstract: We consider the solution of the Schrodinger equation in N dimensions for the infinite N-dimensional spherical potential well. Some aspects of the radial part and the angular part of the wave function are presented and discussed. In particular, the effective potential, orthonormality, energy eigenvalues and the degeneracy are investigated. Thus the role of the topological structure of the configuration space of a physical system on the quantum nature of an observable of the system is emphasized.

Keyword: Quantum Theory, Quantum Mechanics, Formalism

Introduction

Recently, much attention has been paid to the investigation of quantum systems in N-dimensional space. The interest of workers has been in different areas of physics. Grosche and Steiner (1995) and Fukutaka and Kashiwa (1987) considered path integrals and its quantization for a D-dimensional sphere and curved manifolds. Romeo (1995) studied the dependence of the Wentzel-Kramers-Brillouin (WKB) approximations in connection with the hyperspherical quantum billiards. The theory of zero-range potentials and the generalization of Fermi pseudopotentials to higher dimensions were investigated by Wo'dkiewic (1991). The consideration of the eigenvalue bounds for a class of singular potentials in N-dimensions was accomplished by Hall and Saad (1999). Zeng *et al.,* (1994) worked out the most general algebric transformation between a hydrogen atom and a harmonic oscillator of arbitrary dimension. Some properties of the N-dimensional hydrogen atom were reported by Al-Jaber (1998). Recently, there has been emphasis on variaty of problems: Miller (2000) examined the representations and convergence criteria for N-dimensional lattice sums of generalized hypergeometric functions. Fairlie and Leznov (2000) constructed a general solution of the complex Monge-Ampere equation in a space of arbitrary dimension. The quantization of a free particle on a D-dimensional sphere through the Stuckelberg field-shifting formalism was investigated by Neves and Watzasek (2000). Furthermore, Periwal (1995) proposed a formula for continuing physical correlation functions in higher dimensions without perturbation theory.

It is the purpose of this paper to investigate some aspects of the infinite N-dimensional spherical potential well. The N-dimensional Schrodinger equation is examined. There, we derive an effective potential and give the angular solution. We find the radial part of the wave functions and their orthonormality and also we examine the degeneracy of the energy levels.

The N-dimensional Schrodinger Equation: The eigenvalue equation in N dimensions is

$$\frac{-\hbar^2}{2\mu}\nabla^2\psi(\vec{r})+V(r)\psi(\vec{r})=E\psi(\vec{r}), \qquad (1)$$

where

μ is the reduced mass. The Laplacian operator in polar coordinate system, $(r,\theta_1,\theta_2,....,\theta_{N-2},\varphi)$ of R^N is

$$\nabla^2=r^{1-N}\frac{\partial}{\partial r}\left(r^{N-1}\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\Lambda^2, \qquad (2)$$

where

Λ^2 is a partial differential operator on the unit sphere S^{N-1} represented as (Shimakura N. 1992)

$$\Lambda^2=\sum_{K=1}^{N-2}\left(\prod_{j=1}^{K}\sin\theta_j\right)^{-2}(\sin\theta_K)^{K+3-N}\frac{\partial}{\partial\theta_K}\left(\sin\theta_K^{N-K-1}\frac{\partial}{\partial\theta_K}\right)+\left(\prod_{j=1}^{N-2}\sin\theta_j\right)^{-2}\frac{\partial^2}{\partial\varphi^2}; \qquad (3)$$

Writing the wave function in the form

$$\psi(\vec{r})=R(r)f(\theta_j,\varphi), \qquad (4)$$

reduces equation (1) to two separate equations:

$$\Lambda^2 f+\beta f=0, \qquad (5)$$

$$\frac{d^2R}{dr^2}+\frac{N-1}{r}\frac{dR}{dr}+\left[K^2-\frac{\beta}{r^2}\right]R=0, \qquad (6)$$

where

β is a separation constant whose values are given by (Shimakura N. 1992)

$$\beta=L(L+N-2), \qquad (7)$$

with L=0, 1, 2, ..., and $k^2=2\mu(E-V(r))/\hbar^2$. It is tempting to reduce equation (6) to a one-dimensional Schrodinger equation. This can be a chivied as follows: Let

$$R(r)=r^{\alpha}u(r), \qquad (8)$$

where

α is a constant to be determined. The substitution of equation (8) into equation (6) yields

$$\frac{d^2u}{dr^2}+\frac{(2\alpha+N-1)}{r}\frac{du}{dr}+\frac{\alpha(\alpha+N-2)}{r^2}u+\left[K^2-\frac{\beta}{r^2}\right]u=0. \qquad (9)$$

In order that the second term of equation (4) vanishes, we must have $\alpha=(1-N)/2$ and thus the latter equation becomes

$$\frac{d^2u}{dr^2}+\left[k^2-\frac{\beta}{r^2}-\frac{(N-1)(N-3)}{4r^2}\right]u=0. \qquad (10)$$

Equation (10) is the analogue of the one-dimensional Schrodinger equation if one introduces an effective potential given by

$$V_{eff}(r)=V(r)+\frac{\beta}{r^2}+\frac{(N-1)(N-3)}{4r^2}. \qquad (11)$$

This gives the general form of the effective potential in N dimensions. The second term on the right-hand side is well known as the centrifugal barrier and its origin is well understood (Das and Mellissinos 1988). The third term is an additional repulsive potential (for N > 3), which pushes the particle further away from the origin. The case N = 2 makes the third term an attractive one. In the usual three-dimensional space (N=3), this third term vanishes and one recovers the usual effective potential. The presence of this remarkable additional potential term in $V_{eff}(r)$ is another example of the role of the topological structure of the configuration space of a physical system in the quantum nature of the system. The role of the topology of the configuration space in the behavior of the system has been emphasized over the past decade by AL-Jaber and Henneberger (1990), Ho (1994), Ho *et al*., (1996) and by AL-Jaber (1999).

Now, Let us go back to equation (5), where the solution $f(\theta_j,\varphi)$ are the hyperspherical harmonics, $Y_\ell^{\{m\}}(\theta_1,\theta_2,\ldots\theta_{N-2},\varphi)$, of degree ℓ on the sphere S^{N-1}.

For each non-negative integer ℓ, the number of hyperspherical harmonics is given by Mahta and Normand (1997).

$$n_\ell=\frac{(2\ell+N-2)(\ell+N-3)!}{\ell!(N-2)!} \qquad (12)$$

and are characterized by a set of integers

$m_1,m_2,\ldots m_{N-2}$ with the restrictions

$$\ell\geq m_1\geq m_2\geq\ldots\geq m_{N-3}\geq|m_{N-2}|\geq 0. \qquad (13)$$

The hyperspherical harmonics form an orthonormal set, i.e

$$\int Y_\ell^{*\{m\}}(\Omega)Y_{\ell'}^{\{m'\}}(\Omega)d\Omega=\delta_{\ell\ell'}\delta_{\{m\}\{m'\}}, \qquad (14)$$

and thus they form a standard orthonormal basis of the irreducible representations of the rotation group SO(N) in the space of square integrable functions defined over the surface of the N-dimensional unit sphere with the invariant measure (Shimakura, 1992)

$$d\Omega=\prod_{j=1}^{N-2}(\sin\theta_j)^{N-1-j}d\theta_j d\varphi. \qquad (15)$$

An Infinite N-dimensional Spherical Potential Well: We consider an infinite spherical potential well in N dimensions defined as V(r) = 0, for r < a; V(r)= ∞, for r > a. The differential equation for the radial part of the wave function in the region of the well is given by equation (6) with K now is given by $K^2=2\mu E/\hbar^2$. The solution for the differential equation

$$\frac{d^2f}{dx^2}+\left(\frac{1-2a}{x}\right)\frac{df}{dx}+\left[\left(bcx^{c-1}\right)^2+\frac{a^2-p^2c^2}{x^2}\right]f=0, \qquad (16)$$

is given by (Boas, 1983)

$$f(x)=x^a\left[AJ_p(bx^c)+BN_p(bx^c)\right], \qquad (17)$$

Where J_p and N_p are the ordinary Bessel and Neumann functions respectively, and a, b, c and p are constants. Comparing equations (6) and (16) yields a=(2-N)/2, C=1, b=k, and $p=\ell+(N-2)/2$. Therefore, with the help of equation (17), the solution can be readily written down:

$$R(r)=r^{-(N-2)/2}\left[A\,J_{\ell+(N-2)/2}(kr)+B\,N_{\ell+(N-2)/2}(kr)\right], \qquad (18)$$

where A and B are constants

Since R(r) must be finite at r=0, we see that the $N(kr)$ term in equation (18) must be rejected due to its singular behaviour at the origin. Hence

$$R(r)=\frac{A}{r^{(N-2)/2}}\,J_{\ell+(N-2)/2}(kr). \qquad (19)$$

It is clear that the order of the Bessel function is integer for even N and half-odd integer for odd N. For the usual three-dimensional case, N=3, equation (19) gives the well-known Spherical Bessel functions $j_\ell(kr)$ which is found in most standard quantum mechanics textbooks (Griffits, 1995).

In order to discuss the orthonormality of the radial-part solution, $R(kr)$, we adopt the Sturn-Liouville theory. The differential equation (6) can be written in the form

$$\frac{d}{dr}\left(r^{N-1}\frac{dR}{dr}\right)+\left[k^2r^{N-1}-\beta r^{N-3}\right]R=0 \qquad (20)$$

The Sturn-Liouville differential equation has the form

$$\frac{d}{dx}\left[P(x)\frac{dy}{dx}\right]+\left[Q(x)+\lambda W(x)\right]y=0. \qquad (21)$$

It is well-known (Spiegel, 1981) that two different eigenfunctions $y_i(x)$, and $y_j(x)$, corresponding to different eigenvalues, λ_i and λ_j, respectively, are orthogonal in the interval $a\leq x\leq b$ with respect to the weight function W(x). Comparing equations (20) and (21) yields

$$P(r)=W(r)=r^{N-1},\ Q(r)=-\beta r^{N-3}, \qquad (22)$$

and therefore the eigenfunctions $R(kr)$ are orthogonal in the interval $o \le r \le a$ with espect to the weight function $W(r)=r^{N-1}$. The restriction that $R(kr)$ vanishes at $r=a$ implies that $J_\upsilon(ka)=0$, (23)
where

$\upsilon=\ell+(N-2)/2$. If we let $K_{\upsilon\gamma}$ and $K_{\upsilon\delta}$ be the γ-th and the δ-th roots of J_υ respectively then the orthogonality of the corresponding eigen functions reads

$$\int_o^a R_\upsilon(K_{\upsilon\gamma} r/a) R_\upsilon(K_{\upsilon\delta} r/a) r^{N-1} dr = 0. \qquad (24)$$

The normalization of $R_\upsilon(kr)$ can be obtained if one recalls the normalization of the ordinary Bessel functions which is (Arfken, 1985)

$$\int_o^a J_\upsilon^2(K_{\upsilon\gamma} r/a)\, r dr = \frac{a^2}{2} J_{\upsilon+1}^2(K_{\upsilon\gamma}) \qquad (25)$$

Upon the substitution of equation (19) into equation (25) we get

$$\int_o^a R_\upsilon^2(K_{\upsilon\gamma} r/a)\, r^{N-1} dr = \frac{a^N}{2} R_{\upsilon+1}^2(K_{\upsilon\gamma}), \qquad (26)$$

which is the normalization relation for the radial part of the wave function.
The energy eigenvalues of stationary states are readily obtained with the help of equation (23) which gives

$Ka=K_{\upsilon\gamma}$ and therefore the energy eigenvalues are

$$E_{\upsilon\gamma} = \frac{\hbar^2}{2\mu} K_{\upsilon\gamma}^2 . \qquad (27)$$

It must be clear that (Abramowits and Stegum, 1972) for a given $\gamma . K_{\upsilon\gamma} \rangle K_{\upsilon'\gamma}$ whenever $\upsilon \rangle \upsilon'$ and thus, by equation (27) $E_{\upsilon\gamma} \rangle E_{\upsilon'\gamma}$. This shows that for a given ℓ, the higher the dimension N the higher the energy (remember that $\upsilon=\ell+(N-2)/2$). One should expect this result if he recalls that higher dimension N implies higher degree of freedom and thus higher energy.
Now we turn to the question of degeneracy of the energy levels for the infinite N-dimensional potential well. For any spherically symmetric potential in N dimensions the Schrodinger equation can always be separated into ordinary differential equations, one equation for the radial part and another for the angular part. Solutions for the angular part are the hyperspherical harmonics

$Y_\ell^{\{m\}}$. The potential that we are considering has no other symmetries beyond rotational invariance. Therefore the degeneracies of energy levels are the multiplicities of the hyperspherical harmonics for fixed ℓ. Taking into account all possible values of m, for a given ℓ, which are consistent with equation (13), one finds that the degeneracy (deg) is given by

$$\deg = \sum_{m_1=0}^{\ell} \sum_{m_2=0}^{m_1} \cdots \sum_{-m_{N-3}}^{m_{N-3}} |m_{N-2}| . \qquad (28)$$

This number (deg) is equal to

n_ℓ given in equation (12). For the usual three-dimensional case (N=3), equation (28) reduces to

$$\deg = \sum_{-\ell}^{\ell} |m_1| = 2\ell+1 \qquad (29)$$

which is easily predicted by equation (12). For dimensions greater than three, the degeneracy increases. This shows the effect of the dimension of the problem on the physical behaviour of the system. This again emphasizes the role of the topological structure of a system on its physical behaviour.

Results and Discussion

In this paper, a general form of the effective potential was derived when the equation for the radial part of the wave function is written in the form which is analogous to the one - dimensional Schrodinger equation. This effective potential contains, in addition to the usual three-dimensional centrifugal term, an extra term which is repulsive for $N \rangle 3$ and attractive for $N \langle 3$. In order to discuss the role of the dimension of space on the physical behaviour of the system, we considered an instructive example that is the infine -dimensional spherical potential well. For this system, we found the eigenstates and the energy eigenvalues. The radial part of the wave function, R(r), contains Bessel function whose order is integer for even N and half-odd integer for odd N. Also, the normalization condition for R(r) was derived which has an N dependence. The energy eigenvalues are also found to be dependent on the dimension N. The higher the dimension the higher the energy, this must be so since the degree of freedom increases as the dimension N increases and thus implies an increase in the energy. Finally, the angular part of the wave function was presented, that is the hyperspherical harmonics that form an orthonormal basis of the irreducible representations of he rotation group So(N) in the space of the square integrable functions defined on the surface of the N-dimensional unit sphere.

The degeneracy of an energy level for a given ℓ is given by the number of hyperspherical harmonics given by equation (12) and (28).

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